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Every convex free basic semi-algebraic set has an LMI representation

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Abstract

The (matricial) solution set of a Linear Matrix Inequality (LMI) is a convex free basic open semi-algebraic set. The main theorem of this paper is a converse, each such set arises from some LMI. The result has implications for semi-definite programming and systems engineering as well as for free semi-algebraic geometry.

1. **Introduction**

This article involves noncommutative polynomials, their evaluation on tuples of matrices, and (in the spirit of extending classical semi-algebraic geometry to free algebras) noncommutative polynomial inequalities. Here the focus is on such inequalities whose solution sets are matrix convex.

A recurring theme in related noncommutative settings, such as that of a subspace of C^* algebra [Arv69], [Arv72], [Arv08] or in free probability [Voi], [Voi05] to give two of many examples, is the need to consider the *complete matrix structure* afforded by tensoring with $n \times n$ matrices (as n ranges over all positive integers). The resulting theory of operator algebras, systems, spaces, and matrix convex sets has matured to the point that there are now several excellent books on the subject including [BLM04], [Pau02], [Pis03].

A precise statement of results appears later in the body of this introduction. Here, at the beginning, we give a quick indication of the main theorem starting with basic definitions that will be amplified later in the introduction. A free basic open semi-algebraic set is defined in terms of a symmetric free $\delta \times \delta$ matrix-valued polynomial $p(x_1, \cdots, x_g)$. Such a polynomial is a linear combination of words in freely noncommuting variables $(x_1, \ldots, x_g)$ with coefficients from M_δ, the $\delta \times \delta$ matrices over $\mathbb{R}$. The involution ^{T} on words given by sending a concatenation of letters to the same letters, but in the reverse

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order (for instance $(x_j x_\ell)^T = x_\ell x_j$), extends naturally to such polynomials and p is itself symmetric if $p^T = p$.

Let $\mathbb{S}_n(\mathbb{R}^g)$ denote the set of g-tuples $X = (X_1, \dots, X_g)$ of symmetric $n \times n$ matrices. The polynomial p is naturally evaluated on a tuple $X \in \mathbb{S}_n(\mathbb{R}^g)$ yielding a value $p(X)$ that is a $\delta \times \delta$ block matrix with $n \times n$ matrix entries. Evaluation at X is compatible with the involution since $p^T(X) = p(X)^T$. In particular, if p is symmetric, then $p(X)$ is a symmetric matrix.

Assuming that $p(0)$ is invertible, the invertibility set $\mathcal{D}_p(n)$ of a free symmetric polynomial p in dimension n is the component of 0 of the set

$$\{X \in \mathbb{S}_n(\mathbb{R}^g) \ : \ \ p(X) \text{ is invertible}\}.$$

The invertibility set, $\mathcal{D}_p$, is the sequence of sets $(\mathcal{D}_p(n))$. It is an example of a free basic open semi-algebraic set.

The sequence $\mathcal{D}_p$ is convex if $\mathcal{D}_p(n)$ is convex for each n. When $p = L$ is an affine linear symmetric polynomial with constant term I_δ, the expression $L(X) \succ 0$ is a linear matrix inequality and, as is clear, $\mathcal{D}_L$ is a sequence of convex sets.

The main theorem of this article implies *if $p(0)$ is invertible and $\mathcal{D}_p$ is bounded, then there is an ℓ and an affine linear L of size ℓ with constant term I_ℓ such that $\mathcal{D}_p(n) = \mathcal{D}_L(n)$ for each n if and only if $\mathcal{D}_p$ is convex.*

This result is the free algebra analog of the preposterous statement

A bounded open convex set $\mathcal{C}$ in $\mathbb{R}^n$ with algebraic boundary is a polytope.

Since the proof involves matrix convex sets, it is not surprising that our analysis hinges on the matricial version of the Hahn-Banach Separation Theorem of Effros and Winkler [EW97], which says that given a point x outside a matrix convex set $\mathcal{C}$, there is a monic affine linear matrix-valued polynomial that separates x from $\mathcal{C}$. For a general matrix convex set $\mathcal{C}$, the conclusion is then that there is a collection, likely infinite, of monic affine linear matrix-valued polynomials that cut out $\mathcal{C}$. In the case $\mathcal{C}$ is matrix convex and also semi-algebraic, the challenge, successfully dealt with in this paper, is to prove that there is actually a single monic affine linear matrix-valued polynomial L which defines $\mathcal{C}$; i.e., $\mathcal{D}_L = \mathcal{D}_p$.

The article also contains some further results. For instance, a corollary of the results of Section 8 is that if p satisfies certain irreducibility type hypotheses, in addition to the assumption that $\mathcal{D}_p$ is bounded and matrix convex, then p has degree two. In Section 9, implications for free real algebraic geometry, a recently emerging noncommutative analog of the classical subject, are discussed. Classically projections of semi-algebraic sets are semi-algebraic. A consequence of our main result is that this projection property is false in the free case.

The main result also bears on a free analog of semi-definite programming, a major branch of convex optimization. Fundamental in semi-definite programming is the class of convex sets C that can be represented with an LMI, as is the much more general class consisting of projections of LMI representable sets. Their free analogs behave very differently: the classes of projected LMI representable sets that are free semi-algebraic and LMI representable sets are the same. This is shown in Section 9.6.

The remainder of this introduction contains a precise statement of the main result preceded by the relevant definitions.

1.1. *Free polynomials.* Let g be a positive integer, which is now fixed for the remainder of the paper. Let $\mathcal{P}$ denote the real free algebra of polynomials in the freely noncommuting indeterminates $x = (x_1, \dots, x_g)$. Elements of $\mathcal{P}$ are *free polynomials* or often just *polynomials.* Thus, a free polynomial p is a finite linear combination,

$$p = \sum p_w w, \tag{1.1}$$

of words w in $(x_1, \dots, x_g)$ with coefficients $p_w \in \mathbb{R}$.

There is a natural involution ^{T} on $\mathcal{P}$ given by

$$p^T = \sum p_w w^T, \tag{1.2}$$

where, for a word w,

$$w = x_{j_1} x_{j_2} \cdots x_{j_n} \mapsto w^T = x_{j_n} \cdots x_{j_2} x_{j_1}. \tag{1.3}$$

A polynomial p is symmetric if it is invariant with respect to the involution. In particular, $x_j^T = x_j$, and for this reason the variables are sometimes referred to as symmetric free variables.

1.2. *Evaluations.* Let $\mathbb{S}_n(\mathbb{R}^g)$ denote the set of g-tuples $X = (X_1, \dots, X_g)$ of real symmetric $n \times n$ matrices. Let M_n denote the $n \times n$ matrices with real entries. Each $X \in \mathbb{S}_n(\mathbb{R}^g)$ determines a representation $e_X : \mathcal{P} \to M_n$ by evaluation. Indeed, by linearity, e_X is determined by its action on words where $e_X(\emptyset) = I_n$ and for a nonempty word w as in equation (1.3),

$$e_X(w) = X_{j_1} X_{j_2} \cdots X_{j_n}. \tag{1.4}$$

It is natural to write $p(X)$ instead of the more formal $e_X(p)$.

Note that $p(X)$ respects the involution in the sense that $p^T(X) = p(X)^T$. In particular, if p is symmetric, then so is $p(X)$. Finally, if $\pi : \mathcal{P} \to M_n$ is a representation that respects the involution, then there is an $X \in \mathbb{S}_n(\mathbb{R}^g)$ such that $\pi(p) = p(X)$.

1.3. *Matrix-valued polynomials.* Let $\mathcal{P}^{\delta\times\delta'}$ denote the $\delta\times\delta'$ matrices with entries from $\mathcal{P}$. Because row vectors of polynomials figure prominently in this article, $\mathcal{P}^{1\times\delta}$ is often abbreviated to $\mathcal{P}^{\delta}$.

Evaluation at $X\in\mathbb{S}_n(\mathbb{R}^g)$ naturally extends entrywise to $p\in\mathcal{P}^{\delta\times\delta'}$ with the result, $p(X)$, a $\delta\times\delta'$ block matrix with entries from M_n. Up to unitary equivalence, evaluation at X is conveniently described using tensor product notation by writing p as a finite linear combination

$$p=\sum_w p_w w, \tag{1.5}$$

where now the coefficients p_w are $\delta\times\delta$ matrices (with real entries), and observing that

$$p(X)=\sum p_w\otimes w(X),$$

where $w(X)=e_X(w)$ is given by equation (1.4).

The involution ^{T} naturally extends to $\mathcal{P}^{\delta\times\delta}$ by

$$p^T=\sum_w p_w^T w^T$$

for p given by equation (1.5). A polynomial $p\in\mathcal{P}^{\delta\times\delta}$ is symmetric if $p^T=p$, and in this case $p(X)=p(X)^T$.

A simple method of constructing new matrix-valued polynomials from old ones is from direct sums. For instance, if $p_j\in\mathcal{P}^{\delta_j\times\delta_j}$ for $j=1,2$, then

$$p_1\oplus p_2=\begin{pmatrix} p_1 & 0\\ 0 & p_2\end{pmatrix}\in\mathcal{P}^{(\delta_1+\delta_2)\times(\delta_1+\delta_2)}.$$

1.4. *Invertibility sets.* A *graded set* $\mathcal{S}$ is a sequence $\mathcal{S}=(\mathcal{S}(n))_{n=1}^\infty$ where, for each n, $\mathcal{S}(n)\subset\mathbb{S}_n(\mathbb{R}^g)$. The notation $\mathcal{S}\subset\mathbb{S}(\mathbb{R}^g)$ indicates that $\mathcal{S}$ is a graded set. The *principal component* of $\mathcal{S}$, denoted $pc[\mathcal{S}]$, is the connected component of 0 of $\mathcal{S}$; i.e., the graded set $pc[\mathcal{S}]=(pc[\mathcal{S}(n)])$.

Suppose $p\in\mathcal{P}^{\delta\times\delta}$ is symmetric. In particular, $p(0)$ is a $\delta\times\delta$ symmetric matrix. Assuming that $p(0)$ is invertible, for each positive integer n, let

$$\mathfrak{I}_p(n)=\{X\in\mathbb{S}_n(\mathbb{R}^g):\ p(X)\text{ is invertible}\}\subset\mathbb{S}_n(\mathbb{R}^g),$$

and let $\mathfrak{I}_p$ denote the graded set $(\mathfrak{I}_p(n))_{n=1}^\infty$. The *invertibility set* $\mathcal{D}_p$ of p is the graded set $\mathcal{D}_p=pc[\mathfrak{I}_p]$. In Section 9 the graded set $\mathcal{D}_p$ is interpreted in terms of free semi-algebraic geometry.

Remark 1.1. By a simple affine linear change of variable, the point $0\in\mathbb{R}^g$ can be replaced by $\lambda\in\mathbb{R}^g$. For $m>1$, replacing $0\in\mathbb{S}_m(\mathbb{R}^g)$ by a fixed $\Lambda\in\mathbb{S}_m(\mathbb{R}^g)$ will require an extension of the theory.

Remark 1.2. The graded set $\mathcal{D}_p$ is closed with respect to unitary conjugation and direct sums - see Lemma 5.1 for the precise statement. However, because the matrices involved are symmetric, a property not generally preserved under similarity, $\mathcal{D}_p$ is not a free set in the sense of Voiculescu [Voi] [Voi05].

The graded set $\mathcal{D}_p$ is *convex* if each $\mathcal{D}_p(n)$ is convex (in the usual sense). Similarly, $\mathcal{D}_p$ is *bounded* if there is a constant K such for each n and each $X \in \mathcal{D}_p(n)$, $\|X\| = \sum \|X_j\| \leq K$.

The following list of conditions summarizes the usual assumptions on p.

ASSUMPTION 1.3. *Fix p a $\delta \times \delta$ symmetric matrix of polynomials of degree d in g free variables. Our standard assumptions are*

(i) $p(0)$ *is invertible*;
(ii) $\mathcal{D}_p$ *is bounded*; *and*
(iii) $\mathcal{D}_p$ *is convex.*

1.5. *Monic linear pencils.* A *linear pencil* L is an expression of the form

$$L(x) := A_0 + A_1 x_1 + \cdots + A_g x_g \tag{1.6}$$

where, for some positive integer ℓ, each A_j is an $\ell \times \ell$ symmetric matrix with real entries. (While linear pencil is standard usage, it is a bit of a misnomer. When the constant term A_0 is nonzero, a linear pencil is actually affine linear.) The integer ℓ is the *size* of the pencil. The pencil is monic if $A_0 = I$, in which case L is *a monic linear pencil.*

Since a monic linear pencil (of size ℓ) is an element of $\mathcal{P}^{\ell \times \ell}$, it evaluates at a tuple $X \in \mathbb{S}_n(\mathbb{R}^g)$ as

$$L(X) := I_\ell \otimes I_n + A_1 \otimes X_1 + \cdots + A_g \otimes X_g.$$

For a square matrix A, the notation $A \succ 0$ ($A \succeq 0$) indicates that the symmetric matrix A is positive definite (resp. positive semi-definite). From the form of the monic linear pencil L, it is straightforward to verify that its invertibility set is the sequence

$$(\mathcal{D}_L(n)) = (\{X \in \mathbb{S}_n(\mathbb{R}^g) : \; L(X) \succ 0\})$$

and that each $\mathcal{D}_L(n)$ is convex. Moreover,

$$(\overline{\mathcal{D}_L(n)}) = (\{X \in \mathbb{S}_n(\mathbb{R}^g) : \; L(X) \succeq 0\}.$$

A *Linear Matrix Inequality*, or *LMI* for short, is an expression of the form $L(X) \succ 0$. LMI's figure prominently in many branches of engineering and science. A graded subset $\mathcal{C} = (\mathcal{C}(n))$ of the graded set $\mathbb{S}(\mathbb{R}^g)$ has a (*free*) *LMI representation* if there is a monic linear pencil L such that

$$\mathcal{C} = \mathcal{D}_L.$$

The following is the main theorem of this article. A somewhat stronger version of the result appears later as Theorem 9.5.

THEOREM 1.4. *If p satisfies Assumption* 1.3, *then there is a monic linear pencil L (of finite size) such that $\mathcal{D}_p(n) = \mathcal{D}_L(n)$ for every n; that is, if $p \in \mathcal{P}^{\delta\times\delta}$ is symmetric, $p(0)$ is invertible, and $\mathcal{D}_p$ is bounded, then $\mathcal{D}_p$ is convex if and only if the graded set $\mathcal{D}_p$ has an* LMI *representation.*

Results needed for the proof of Theorem 1.4 occupy the paper up through Section 6. The proof of the theorem itself appears in Section 7. That section also gives a bound, depending only upon the degree d, the number of variables g, and the (matrix) size δ of p, on the size of the linear pencil L needed to represent $\mathcal{D}_p$. Section 8 refines the main theorem by adding irreducibility type hypotheses on p and concluding that p has degree two. Implications for free real algebraic geometry and semi-definite programming appear in Section 9.

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2. Preliminaries

From now through Section 7, fix a polynomial p satisfying the conditions of Assumption 1.3. Thus amongst other things, p is $\delta \times \delta$ matrix-valued, has degree d, and is a polynomial in g freely noncommuting variables.

This section presents two basic facts for future use. The following lemma gives a useful criterion for containment in the closure $\overline{\mathcal{D}}_p = (\overline{\mathcal{D}_p(n)})$ of the graded set $\mathcal{D}_p$.

LEMMA 2.1. *Suppose $p \in \mathcal{P}^{\delta\times\delta}$ satisfies the conditions of Assumption* 1.3 *and n is a positive integer. If $X \in \mathbb{S}_n(\mathbb{R}^g)$, then $X \in \overline{\mathcal{D}}_p(n)$ if and only if $tX \in \mathcal{D}_p(n)$ for all $0 \le t < 1$.*

Proof. First suppose that $X \in \overline{\mathcal{D}}_p(n)$. Since $\mathcal{D}_p(n)$ is convex, so is $\overline{\mathcal{D}}_p(n)$. Further, $\mathcal{D}_p(n)$ contains $0 \in \mathbb{S}_n(\mathbb{R}^g)$. Thus, $tX \in \overline{\mathcal{D}}_p(n)$ for $0 \le t \le 1$. Moreover, there are only finitely many $0 \le s \le 1$ such that $p(sX)$ is not invertible because $p(0)$ is invertible and p is a polynomial. Indeed, $p(sX)$ is invertible if and only if the nonzero polynomial $q(t) = \det(p(tX))$ is not zero at s. If $0 \le t < 1$ and $p(tX)$ is invertible, then $tX \in \mathfrak{I}_p(n)$. To see that tX is in fact in $\mathcal{D}_p(n)$, we argue by contradiction. Accordingly, suppose $tX \notin \mathcal{D}_p(n)$. In this case, since $\mathfrak{I}_p(n)$ is both open and the disjoint union of its connected components, tX is contained in some open set that does not meet $\mathcal{D}_p(n)$. Thus, $tX \notin \overline{\mathcal{D}}_p(n)$, a contradiction. Now $tX \in \mathcal{D}_p(n)$, and since $\mathcal{D}_p(n)$ is convex, $sX \in \mathcal{D}_p(n)$ for $0 \le s \le t$. Choosing a sequence $0 < t_n < 1$ converging to 1 such that $p(t_nX)$ is invertible, it now follows that $sX \in \mathcal{D}_p(n)$ for $0 \le s < 1$.

The converse is evident. $\square$

LEMMA 2.2. *Let $\mathcal{C}=(\mathcal{C}(n))$ be a graded set with $\mathcal{C}(n)\subset\mathbb{S}_n(\mathbb{R}^g)$ for each n. If each $\mathcal{C}(n)$ is open and if L is a monic linear pencil, then L is positive definite on each $\mathcal{C}(n)$ if and only if L is positive semi-definite on each $\mathcal{C}(n)$.*

Proof. Suppose L is positive semi-definite on $\mathcal{C}(n)$. If L is not positive definite on $\mathcal{C}(n)$, then there is an an $X\in\mathcal{C}(n)$ such that $L(X)\succeq 0$ and $L(X)$ has a kernel. In particular, there is a unit vector v such that $L(X)v=0$. Let $q(t)=\langle L(tX)v,v\rangle$. Thus q is affine linear and $q(0)=1$, whereas $q(1)=0$. Hence $q(t)<0$ for $t>1$ and thus $L(tX)\not\succeq 0$ for $t>1$. On the other hand, since $\mathcal{C}(n)$ is open and $X\in\mathcal{C}(n)$, there is $t>1$ such that $tX\in\mathcal{C}(n)$, which contradicts $L(tX)\succeq 0$. □

3. Dominating points and the boundaries of $\mathcal{D}_p$

There are two notions, both important for what follows, of the boundary of the graded set $\mathcal{D}_p$. The (*topological*) *boundary* of $\mathcal{D}_p$, denoted $\partial\mathcal{D}_p$, is the graded set $(\partial\mathcal{D}_p(n))$ where $\partial\mathcal{D}_p(n)$ is the usual topological boundary of $\mathcal{D}_p(n)$. Let $\widehat{\partial\mathcal{D}}_p(n)$ denote the set of pairs (X,v) where $X\in\partial\mathcal{D}_p(n)$, the vector v is in $\mathbb{R}^\delta\otimes\mathbb{R}^n$, and $p(X)v=0$. The assumption $v\neq 0$ will often be implicit. The graded set $\widehat{\partial\mathcal{D}}_p=(\widehat{\partial\mathcal{D}}_p(n))$ is the *detailed boundary* of $\mathcal{D}_p$. The use of the term graded set for $\widehat{\partial\mathcal{D}}_p$, while technically different from the use of the term graded set defined earlier, should cause no confusion.

Given $(X^j,v^j)\in\mathbb{S}_{n_j}(\mathbb{R}^g)\times(\mathbb{R}^\delta\otimes\mathbb{R}^{n_j})$, for $j=1,2$, let

$$\oplus_{j=1}^2(X^j,v^j)=\left(\begin{pmatrix}X^1 & 0\\ 0 & X^2\end{pmatrix},\begin{pmatrix}v^1\\ v^2\end{pmatrix}\right).$$

This notion of direct sum clearly extends to a finite list (X^j,v^j), $j=1,2,\dots,s$. A graded set $S=(S(n))$ where $S(n)\subset\mathbb{S}_n(\mathbb{R}^g)\times(\mathbb{R}^\delta\otimes\mathbb{R}^n)$ *respects direct sums* if $(X^j,v^j)\in S(n_j)$, for $j=1,2,\dots,s$, implies $\oplus_1^s(X^j,v^j)\in S(n)$, where $n=\sum n_j$. It is evident that the graded set $\widehat{\partial\mathcal{D}}_p=(\widehat{\partial\mathcal{D}}_p(n))$ respects direct sums.

Let $\mathcal{P}_d^\delta$ denote the $1\times\delta$ (row) matrices with entries polynomials of degree at most d. If $X\in\mathbb{S}_n(\mathbb{R}^g)$ and $q\in\mathcal{P}_d^\delta$, then $q(X)$ is a linear mapping from $\mathbb{R}^\delta\otimes\mathbb{R}^n$ to $\mathbb{R}^n$. Hence if $(X,v)\in\widehat{\partial\mathcal{D}}_p(n)$, then $q(X)v$ is defined. Let $T=(T(n))$ denote a nonempty graded subset of the graded set $\widehat{\partial\mathcal{D}}_p$. A point $(X,v)\in\widehat{\partial\mathcal{D}}_p(m)$ is a *dominating point of* T if, for a given $q\in\mathcal{P}_d^\delta$, $q(X)v=0$ implies that $q(Y)w=0$ for every n and $(Y,w)\in T(n)$; i.e., if q vanishes at (X,v), then q vanishes on all of T. Let T_* denote the dominating points of T. Note T_* need not be contained in T.

Given a graded subset $T=(T(n))_{n=1}^\infty$ of the graded set $\widehat{\partial\mathcal{D}}_p$, let

$$\mathcal{I}(T)=\{q\in\mathcal{P}_d^\delta:\ q(X)v=0\ \text{ for all } (X,v)\in T\}\subset\mathcal{P}_d^\delta.$$

In the special case that $T=\{(X,v)\}$ is a singleton (so there is an m such that $T(m)$ has one element and $T(n)$ is the empty set for all other n), the notation

$\mathcal{I}(X,v)$ is used in place of the more cumbersome $\mathcal{I}(\{(X,v)\})$. Note that, in the case $\delta = 1$, if not for the degree d restriction, the subspace $\mathcal{I}(T)$ would be a left ideal in $\mathcal{P}$. In any case, $\mathcal{I}(T)$ is a subspace of $\mathcal{P}_d^\delta$. Note that $\mathcal{I}(T)$ contains each row of p and is thus not the zero subspace.

The following lemma follows readily from the definitions.

LEMMA 3.1. *Let $T = (T(n))$ be a nonempty graded subset of the graded set $\widehat{\partial\mathcal{D}}_p$. The point $(X,v) \in \widehat{\partial\mathcal{D}}_p$ is a dominating point of T if and only if*

$$\mathcal{I}(X,v) \subset \mathcal{I}(T).$$

On the other hand, if $(X,v) \in T(n)$, then

$$\mathcal{I}(T) \subset \mathcal{I}(X,v).$$

Thus, if $(X,v) \in T(n) \cap T_*(n)$, then

$$\mathcal{I}(X,v) = \mathcal{I}(T).$$

Given graded subsets $A = (A(n))$ and $B = (B(n))$ of $\widehat{\partial\mathcal{D}}_p$, the *intersection* of A and B, denoted $A \cap B$, is the graded set $(A(n) \cap B(n))$. Similarly, A is *nonempty* if there is an m so that $A(m)$ is nonempty. The following two lemmas are key facts about dominating points for graded sets that respect direct sums.

LEMMA 3.2. *Suppose $S = (S(n))$ is a nonempty graded subset of the graded set $\widehat{\partial\mathcal{D}}_p$. If S respects direct sums, then there is an m and a $(X,v) \in S(m)$ such that*

$$\mathcal{I}(X,v) = \mathcal{I}(S). \tag{3.1}$$

Hence $S \cap S_*$ is nonempty.

Proof. First note that

$$\mathcal{I}(S) = \bigcap \{\, \mathcal{I}(Y,w) : \ (Y,w) \in S\}.$$

Thus, since each $\mathcal{I}(Y,w)$ is a subspace of the finite dimensional vector space $\mathcal{P}_d^\delta$, there exists an s and $(Y_j, w_j) \in S(n_j)$ for $j = 1, \dots, s$ such that

$$\mathcal{I}(S) = \cap_{j=1}^s \mathcal{I}(Y_j, w_j).$$

Let $(X,v) = \oplus(Y_j, w_j)$. Then $(X,v) \in S(m)$, where $m = \sum n_j$, and

$$\mathcal{I}(X,v) = \cap_{j=1}^s \mathcal{I}(Y_j, w_j) = \mathcal{I}(S). \tag{3.2}$$

□

LEMMA 3.3. *Suppose $S = (S(n))$ is a graded subset of the graded set $\widehat{\partial\mathcal{D}}_p$ that respects direct sums, and suppose $q \in \mathcal{P}_d^\delta$. If $(X,v) \in S(n) \cap S_*(n)$ and $(Y,w) \in S(m) \cap S_*(m)$, then $q(X)v = 0$ if and only if $q(Y)w = 0$; that is, q either vanishes on the whole graded set $S \cap S_* = (S(n) \cap S_*(n))$ or none of $S \cap S_*$.*

Proof. From Lemma 3.1 (twice),

$$I(X,v) = I(S) = I(Y,w). \qquad \square$$

This section closes with the following observation. A graded subset $Z = (Z(n))$ of the graded set $\widehat{\partial\mathcal{D}_p}$ *respects simultaneous unitary conjugation* if, for each n, $(X,v) \in Z(n)$ and $n \times n$ unitary U,

$$U^T(X,v)U := ((U^T X_1 U, \dots, U^T X_g U), U^T v) \in Z(n). \tag{3.3}$$

Lemma 3.4. *If I is a subset of $\mathcal{P}_d^\delta$, then the graded set $\mathcal{Z}(I) = (\mathcal{Z}(I)(n))$ defined by*

$$\mathcal{Z}(I)(n) = \{(X,v) \in \widehat{\partial\mathcal{D}_p}(n) : f(X)v = 0 \textit{ for all } f \in I\}$$

respects both direct sums and unitary conjugations.

Further, if $I \subset J \subset \mathcal{P}_d^\delta$, then $\mathcal{Z}(I)(n) \supset \mathcal{Z}(J)(n)$ for every n; that is, $\mathcal{Z}(I) \supset \mathcal{Z}(J)$.

Proof. The first statement is evident if I contains a single $q \in \mathcal{P}_d^\delta$. The general result follows by observing that the properties of respecting direct sums and unitary conjugations are preserved under (termwise) intersection of graded sets.

The statement about inclusions is readily verified. $\square$

4. Closure with respect to a subspace of polynomials

In this section a canonical closure operation on graded subsets $W = (W(n))$ of the graded set $\widehat{\partial\mathcal{D}_p}$ is introduced and its properties developed. Recall that the positive integers d, δ, and g have all been fixed (by p) and that $\mathcal{P}_d^\delta$ denotes the $1 \times \delta$ matrices whose entries are free polynomials of degree at most d in g freely noncommuting symmetric variables.

The $\mathcal{P}_d^\delta$*-closure* of a nonempty graded subset $W = (W(n))$ of $\widehat{\partial\mathcal{D}_p}$ is the graded set $W_z = (W_z(n))$ where,

$$W_z(n) := \{(X,v) \in \widehat{\partial\mathcal{D}_p}(n) : f(X)v = 0 \text{ for every } f \in \mathcal{I}(W)\}.$$

In particular, to say W is $\mathcal{P}_d^\delta$*-closed* means $W_z = W$.

Lemma 4.1. *Let $W = (W(n))$ denote a nonempty graded subset of $\widehat{\partial\mathcal{D}_p}$.*

(i) *In the notation of Lemma 3.4, $W_z = \mathcal{Z}(\mathcal{I}(W))$;*
(ii) *If $(X,v) \in \widehat{\partial\mathcal{D}_p}(n)$, then $(X,v) \in W_z(n)$ if and only if $\mathcal{I}(X,v) \supset \mathcal{I}(W)$;*
(iii) *$\mathcal{I}(W) = \mathcal{I}(W_z)$; and*
(iv) *If $U = (U(n))$ is a graded subset of $\widehat{\partial\mathcal{D}_p}$ and $\mathcal{I}(U) = \mathcal{I}(W)$, then $U \subset W_z$; that is, $U(n) \subset W_z(n)$ for every n.*

Note that item (iv) says that W_z is the largest graded subset of $\widehat{\partial\mathcal{D}_p}$ such that $I(W_z) = I(W)$.

Proof. The first item is evident. To prove item (ii), suppose $(X, v) \in W_z(n)$. If $q \in \mathcal{I}(W)$, then $q(X)v = 0$ and hence $q \in \mathcal{I}(X, v)$. Thus, $\mathcal{I}(W) \subset \mathcal{I}(X, v)$. Conversely, suppose $(X, v) \in \widehat{\partial\mathcal{D}}_p(n)$ and $\mathcal{I}(X, v) \supset \mathcal{I}(W)$. If $q \in \mathcal{I}(W)$, then $q \in \mathcal{I}(X, v)$ and hence $q(X)v = 0$. Hence $(X, v) \in W_z(n)$.

Since $(X, v) \in W_z$ implies $\mathcal{I}(X, v) \supset \mathcal{I}(W)$, it follows that $\mathcal{I}(W_z) \supset \mathcal{I}(W)$. On the other hand, since $W \subset W_z$, the inclusion $\mathcal{I}(W) \supset \mathcal{I}(W_z)$ and the equality $\mathcal{I}(W) = \mathcal{I}(W_z)$ follows.

Finally, suppose $\mathcal{I}(U) = \mathcal{I}(W)$ and let $(X, v) \in U$ be given. If $q \in \mathcal{I}(W)$, then $q \in \mathcal{I}(U)$ and hence $q(X)v = 0$. Thus, $(X, v) \in W_z$ and hence $U \subset W_z$. $\square$

The following lemma collects basic facts about the $\mathcal{P}_d^\delta$-closure operation. The statement and proof extensively use the following conventions. Given graded subsets $A = (A(n))$ and $B = (B(n))$ of the graded set $\widehat{\partial\mathcal{D}}_p$, the notation $A \subset B$ means $A(n) \subset B(n)$ for each n. Similarly, the notation $A \subsetneq B$ means $A \subset B$ and there is an m so that $A(m) \subsetneq B(m)$.

LEMMA 4.2. *Suppose $\widehat{\partial\mathcal{D}}_p \supset A, B$ are nonempty graded sets that respect direct sums.*

(i) $A \subset A_z$;
(ii) *If $A \supset B$, then $\mathcal{I}(A) \subset \mathcal{I}(B)$;*
(iii) *If $\mathcal{I}(A) \subset \mathcal{I}(B)$, then $A_z \supset B_z \supset B$;*
(iv) *If $B \subset A$, then $B_z \subset A_z$;*
(v) *If B is $\mathcal{P}_d^\delta$-closed and $B \subsetneq A$, then $\mathcal{I}(A) \subsetneq \mathcal{I}(B)$;*
(vi) *If $A_1 \supset A_2 \supset \cdots$ is a decreasing sequence of nonempty $\mathcal{P}_d^\delta$-closed sets, then there is an m such that $A_m = A_\ell$ for all $\ell \geq m$; and*
(vii) *A nonempty collection $\mathfrak{T}$ of nonempty $\mathcal{P}_d^\delta$-closed subsets of $\widehat{\partial\mathcal{D}}_p$ contains a minimal element; i.e., there exists a set $T \in \mathfrak{T}$ such that if $A \subset T$ and $A \in \mathfrak{T}$, then $A = T$.*

Proof. The first four items are obvious. To prove (v), note that by (ii), $\mathcal{I}(A) \subset \mathcal{I}(B)$. On the other hand, if $\mathcal{I}(A) = \mathcal{I}(B)$, then by (iii), $A_z = B_z$. But then, because B is $\mathcal{P}_d^\delta$ closed,

$$B_z = B \subsetneq A \subset A_z = B_z,$$

a contradiction.

Item (vi) holds because, by (v), $\mathcal{I}(A_1) \subset \mathcal{I}(A_2) \subset \cdots$ is an increasing nest of subspaces of the finite dimensional vector space $\mathcal{P}_d^\delta$. Thus there is an m such that $\mathcal{I}(A_\ell) = \mathcal{I}(A_m)$ for all $\ell \geq m$. Using (iii) twice and the fact that each A_ℓ is $\mathcal{P}_d^\delta$-closed, it follows that $A_\ell = A_m$ for $\ell \geq m$.

To prove (vii), choose $A_1 \in \mathfrak{T}$. If A_1 is not minimal, then there exists $A_2 \in \mathfrak{T}$ such that $A_1 \supsetneq A_2$. Continuing in this fashion, eventually produces a minimal set T as the alternative is a nested strictly decreasing sequence

$$A_1 \supsetneq A_2 \supsetneq A_3 \supsetneq \cdots$$

from $\mathfrak{T}$, which contradicts (vi). $\square$

Facts about the relation between dominating points and $\mathcal{P}_d$-closures are collected in the next lemma. Recall the characterization of dominating points given in Lemma 3.1. If A is a graded subset of $\widehat{\partial\mathcal{D}_p}$, let A_* denote the graded set $A_* = (A_*(n))$. Recall, if B is also a graded set, then $A \cap B$ is the graded set $(A(n) \cap B(n))$.

LEMMA 4.3. *Suppose $\widehat{\partial\mathcal{D}_p} \supset A, B$ are nonempty graded sets that respect direct sums.*

(i) *If $A \supset B$, then $A_* \subset B_*$;*
(ii) $A_* = (A_z)_*$;
(iii) $B \cap B_*$ *is nonempty*;
(iv) *Moreover*,

$$B \cap B_* \subset \{(X,v) \in \widehat{\partial\mathcal{D}_p} : \mathcal{I}(X,v) = \mathcal{I}(B)\} \text{ and}; \tag{4.1}$$

(v) *If A is $\mathcal{P}_d^\delta$ closed, then*

$$A \cap A_* = \{(X,v) \in \widehat{\partial\mathcal{D}_p} : \mathcal{I}(X,v) = \mathcal{I}(A)\}.$$

Hence for any B,

$$B_z \cap B_* = \{(X,v) \in \widehat{\partial\mathcal{D}_p} : \mathcal{I}(X,v) = \mathcal{I}(B)\}.$$

Remark 4.4. Note that (iii) is Lemma 3.2 and (iv) is part of Lemma 3.1.

Proof. To prove item (i) observe, if $(X,v) \in A_*(n)$, then, by Lemma 3.1 and Lemma 4.2(ii), $\mathcal{I}(X,v) \subset \mathcal{I}(A) \subset \mathcal{I}(B)$. Thus, by another application of Lemma 3.1, $(X,v) \in B_*(n)$.

By Lemma 4.2(i), $A \subset A_z$. Thus, by part (i) of this lemma, $A_* \supset (A_z)_*$. On the other hand, if $(X,v) \in A_*(n)$, then, in view of Lemma 4.1(iii),

$$\mathcal{I}(X,v) \subset \mathcal{I}(A) = \mathcal{I}(A_z)$$

and thus $(X,v) \in (A_z)_*(n)$. Hence $A_* \subset (A_z)_*$ and item (ii) is proved.

It remains to prove item (v). One inclusion follows from (iv). To prove the other inclusion, suppose A is $\mathcal{P}_d^\delta$-closed, $(X,v) \in \widehat{\partial\mathcal{D}_p}$, and $\mathcal{I}(X,v) = \mathcal{I}(A)$. Since $\mathcal{I}(X,v) \supset \mathcal{I}(A)$ and A is $\mathcal{P}_d^\delta$-closed, item (ii) of Lemma 4.1 implies $(X,v) \in A$. On the other hand, $(X,v) \in A_*$ since $\mathcal{I}(X,v) \subset \mathcal{I}(A)$. Thus the reverse inclusion holds and the proof of the first part of (v) is complete. The second part of (v) follows from the first part and item (iii) of Lemma 4.1. $\square$

For a monic linear pencil L, let $i(L)$ denote the graded subset $(i(L)(n))$ of the graded set $\widehat{\partial\mathcal{D}_p}$ defined by

$$i(L)(n) := \{(Y,w) \in \widehat{\partial\mathcal{D}_p}(n) : \ L(Y) \text{ is invertible}\}.$$

If S is a graded subset of $\widehat{\partial\mathcal{D}_p}$, then L is said to be *singular on* S if $L(X)$ is not invertible for each n and $(X,v) \in S(n)$; i.e., if $S(n) \cap i(L)(n)$ is empty for each n.

Proposition 4.5. *Let $S = (S(n))$ be a nonempty graded subset of the graded set $\widehat{\partial\mathcal{D}_p}$. Suppose S respects direct sums and L is a monic linear pencil. If*

(i) *L is singular on S_*; and*
(ii) *$\emptyset \neq i(L) \subset S$,*

then $i(L)_z$ is properly contained in S_z; i.e., there is an m such that

$$i(L)_z(m) \subsetneq S_z(m).$$

Proof. Item (ii) and Lemma 4.2(iv) imply $i(L)_z \subset S_z$. Arguing by contradiction, suppose that $i(L)_z = S_z$. Then, from Lemma 4.3(ii) (twice),

$$i(L) \cap i(L)_* = i(L) \cap (i(L)_z)_* = i(L) \cap (S_z)_* = i(L) \cap S_*.$$

On the other hand, since $i(L) = (i(L)(n))$ is a nonempty graded subset of $\widehat{\partial\mathcal{D}_p}$ that respects direct sums, by Lemma 3.2, there is an m and an $(X,v) \in i(L)(m) \cap i(L)_*(m)$. Hence there is an $(X,v) \in i(L)(m) \cap S_*(m)$. But then $L(X)$ is invertible, since $(X,v) \in i(L)$, and on the other hand, by (i), $L(X)$ is singular because $(X,v) \in S_*(m)$. This contradiction proves the indicated inclusion is proper. $\square$

5. **Convexity and the invertibility set**

This section contains proofs of two facts about the convex graded set $\mathcal{D}_p$. First, it is in fact an open matrix convex set (see Definition 5.4 below), and second, membership in $\mathcal{D}_p$ and its boundary is determined by compressions to subspaces of dimension at most $\nu = \delta \sum_0^d g^j$. (Recall, p is $\delta \times \delta$ matrix-valued, d is the degree of p, and g is the number of variables.)

5.1. *Matrix convexity.* A graded subset $S = (S(n))$ of $\mathbb{S}(\mathbb{R}^g)$ *respects simultaneous unitary conjugation* if for each n, $X \in S(n)$ and each $n \times n$ unitary matrix,

$$U^T XU = (U^T X_1 U, \ldots, U^T X_g U) \in S(n).$$

This is analogous to (3.3). The following lemma applies to any $\mathcal{D}_q$, whether convex or not. The second item has already been used repeatedly.

Lemma 5.1. *Suppose $q \in \mathcal{P}^{r\times r}$ is symmetric and $q(0)$ is invertible.*

(i) *The graded set* $\mathcal{D}_q$ *respects simultaneous unitary conjugation*; *and*
(ii) $\mathcal{D}_q$ *respects direct sums.*

Proof. The first item follows from the fact that $q(U^TXU) = U^Tq(X)U$ and the second from $q(X \oplus Y) = q(X) \oplus q(Y)$. □

Recall, by definition, $\mathcal{D}_p = (\mathcal{D}_p(n))$ is convex if each $\mathcal{D}_p(n)$ is convex.

LEMMA 5.2. *If* $\mathcal{D}_p$ *is convex,* $X \in \mathbb{S}_n(\mathbb{R}^g)$, $Y \in \mathbb{S}_m(\mathbb{R}^g)$, *and* $X \oplus Y \in \mathcal{D}_p(n+m)$, *then* $X \in \mathcal{D}_p(n)$ *and* $Y \in \mathcal{D}_p(m)$.

Proof. Let $Z = X \oplus Y \in \mathcal{D}_p(n+m)$. By convexity, $tZ \in \mathcal{D}_p(n+m)$ for $0 \le t \le 1$. It follows that $p(tX)$ is invertible for $0 \le t \le 1$ and so there is a path from 0 to X lying in $\mathcal{D}_p(n)$. Thus $X \in \mathcal{D}_p(n)$. Likewise for Y. □

Remark 5.3. It is not clear if Lemma 5.2 remains true with the weaker hypothesis that the closure of $\mathcal{D}_p$ is convex.

Definition 5.4. For the present purposes a graded set $\mathcal{C} = (\mathcal{C}(n))$, where each $\mathcal{C}(n) \subset \mathbb{S}_n(\mathbb{R}^g)$, is a bounded open *matrix convex* set if

(i) each $\mathcal{C}(m)$ is open and contains $0 = (0, \dots, 0) \in \mathbb{S}_m(\mathbb{R}^g)$;
(ii) $\mathcal{C}$ respects direct sums;
(iii) $\mathcal{C}$ *respects simultaneous conjugation with contractions*: if $Y \in \mathcal{C}(m)$ and F is an $m \times k$ contraction, then

$$F^TYF = (F^TY_1F, \dots, F^TY_gF) \in \mathcal{C}(k); \text{ and}$$

(iv) each $\mathcal{C}(m)$ is convex and bounded.

There are some harmless redundancies in the conditions above. It is easy to see that the convexity of $\mathcal{C}(m)$ actually follows from items (ii) and (iii). Indeed, given $X, Y \in \mathcal{C}(n)$, choose F to be the $2n \times n$ matrix

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n \\ I_n \end{pmatrix}$$

and note that

$$\frac{X_j + Y_j}{2} = F^T \begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} F \qquad \text{for each } j.$$

Similarly, if it assumed that $\mathcal{C}$ is not empty, then that $0 \in \mathcal{C}(n)$ for all n follows from (iii) by choosing $F = 0$.

An immediate consequence of item (iii) is, if $X \in \mathbb{S}_n(\mathbb{R}^g)$, $Y \in \mathbb{S}_m(\mathbb{R}^g)$ and $X \oplus Y \in \mathcal{C}(n+m)$, then $Y \in \mathcal{C}(m)$.

THEOREM 5.5. *If* p *satisfies the conditions of Assumption* 1.3, *then* $\mathcal{D}_p$ *is a bounded open matrix convex set.*

Proof. That $\mathcal{D}_p$ is closed with respect to direct sums is part of Lemma 5.1 (and does not depend upon convexity or boundedness).

To prove that $\mathcal{D}_p$ is closed with respect to simultaneous conjugation by contractions, suppose that $X \in \mathcal{D}_p(n)$ and F is a given $n \times k$ contraction. Let U denote the Julia matrix (of F),

$$U = \begin{pmatrix} F & (I_n - FF^T)^{\frac{1}{2}} \\ -(I_k - F^T F)^{\frac{1}{2}} & F^T \end{pmatrix}.$$

Routine calculations show U is unitary.

Let 0 denote the g-tuple of zero matrices of size $k \times k$. Then, since $X \in \mathcal{D}_p(n)$ and $0 \in \mathcal{D}_p(k)$, the direct sum $X \oplus 0$ is in $\mathcal{D}_p(n+k)$. Since $\mathcal{D}_p(n+k)$ is closed with respect to unitary conjugation, both the g-tuples of matrices

$$Y = U^T \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} U,$$

$$Z = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} Y \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

are in $\mathcal{D}_p(n+k)$. Using the convexity assumption on $\mathcal{D}_p(n+k)$,

$$\frac{1}{2}(Y+Z) = \begin{pmatrix} F^T X F & 0 \\ 0 & (I - FF^T)^{\frac{1}{2}} X (I - FF^T)^{\frac{1}{2}} \end{pmatrix}$$

is in $\mathcal{D}_p(n+k)$. An application of Lemma 5.2 implies $F^T X F \in \mathcal{D}_p(n)$.

By hypothesis, $\mathcal{D}_p$ is bounded. $\square$

5.2. *Compressions.* Recall $\mathcal{P}_d^\delta$ denotes the $1 \times \delta$ matrices whose entries are free polynomials of degree at most d in g freely noncommuting variables. Given $(X, v) \in \mathbb{S}_n(\mathbb{R}^g) \times (\mathbb{R}^\delta \otimes \mathbb{R}^n)$, define the subspace $\mathcal{M} = \mathcal{M}(X, v)$ of $\mathbb{R}^n$ by

$$\mathcal{M} := \{q(X)v : \ q \in \mathcal{P}_d^\delta\} \subset \mathbb{R}^n. \tag{5.1}$$

Explicitly, v is a column vector of length δ with entries from $\mathbb{R}^n$ and

$$q(X)v = \begin{pmatrix} q_1(X) \dots q_\delta(X) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_\delta \end{pmatrix} = \sum q_j(X) v_j, \tag{5.2}$$

where each q_j is free polynomial of degree at most d.

Let $P_{\mathcal{M}}$ denote the projection of $\mathbb{R}^n$ onto $\mathcal{M}$. Consistent with previous usage, the notation $P_{\mathcal{M}} X|_{\mathcal{M}}$ is shorthand for $(P_{\mathcal{M}} X_1|_{\mathcal{M}}, \dots, P_{\mathcal{M}} X_g|_{\mathcal{M}})$. The integer $\nu = \delta \sum_{j=0}^d g^j$, the dimension of the vector space $\mathcal{P}_d^\delta$, is an upper bound for the dimension of the vector space $\mathcal{M}$.

Lemma 5.6. *Suppose p satisfies the hypotheses of Assumption 1.3 and n is a positive integer. If $(X, v) \in \widehat{\partial \mathcal{D}_p}(n)$ and μ is the dimension of $\mathcal{M} = \mathcal{M}(X, v)$, then $(P_{\mathcal{M}} X|_{\mathcal{M}}, v) \in \widehat{\partial \mathcal{D}_p}(\mu)$. In fact, $t P_{\mathcal{M}} X|_{\mathcal{M}} \in \mathcal{D}_p(\mu)$ for $0 \leq t < 1$ and $p(P_{\mathcal{M}} X|_{\mathcal{M}})v = 0$.*

Proof. From Lemma 2.1, $tX \in \mathcal{D}_p(n)$ for $0 \leq t < 1$. Let V denote the inclusion of $\mathcal{M}$ into $\mathbb{R}^n$. Since V is a contraction and, by Theorem 5.5, $\mathcal{D}_p$ is a (open) matrix convex set, $t P_{\mathcal{M}} X|_{\mathcal{M}} = V^T t X V \in \mathcal{D}_p(\mu)$.

Writing v as in equation (5.2), for any word w of length at most d and any $1 \leq j \leq \delta$,

$$w(P_{\mathcal{M}} X|_{\mathcal{M}})v_j = P_{\mathcal{M}} w(X)|_{\mathcal{M}} v_j = P_{\mathcal{M}} w(X) v_j.$$

Hence,

$$p(P_{\mathcal{M}} X|_{\mathcal{M}})v = (I_\delta \otimes P_{\mathcal{M}}) p(X) v = 0. \qquad \square$$

6. Separating monic linear pencils

This section develops a refinement of the matricial Hahn-Banach Separation Theorem of Effros-Winkler for the graded set $\mathcal{D}_p$, Proposition 6.8 in Section 6.3. A version of the Effros-Winkler Separation Theorem is the topic of the first subsection.

6.1. *A version of the Effros-Winkler theorem.* This subsection contains a proof of the separation theorem of Effros and Winkler [EW97] in the special case of certain matrix convex subsets of $\mathbb{S}(\mathbb{R}^g) = (\mathbb{S}_n(\mathbb{R}^g))_{n=1}^\infty$. The specialization makes the proof of Proposition 6.4, which is applied in the following subsection, simpler than that of the strictly more general version in [EW97]. On the other hand, Proposition 6.4 is not explicitly covered by the results in [EW97].

Given a positive integer n, let $\mathcal{T}_n$ denote the positive semi-definite $n \times n$ matrices (with real entries) of trace one. Each $T \in \mathcal{T}_n$ corresponds to a state on M_n, the $n \times n$ matrices, via the trace

$$M_n \ni A \mapsto \operatorname{tr}(AT).$$

Note that $\mathcal{T}_n$ is a convex, compact subset of $\mathbb{S}_n$, the symmetric $n \times n$ matrices.

The following lemma is a version of Lemma 5.2 from [EW]. An affine linear mapping $f : \mathbb{S}_n \to \mathbb{R}$ is a function of the form $f(x) = a_f + \lambda_f(x)$, where λ_f is linear and $a_f \in \mathbb{R}$.

Lemma 6.1. *Suppose $\mathcal{F}$ is a convex set of affine linear mappings $f : \mathbb{S}_n \to \mathbb{R}$. If for each $f \in \mathcal{F}$ there is a $T \in \mathcal{T}_n$ such that $f(T) \geq 0$, then there is a $\mathfrak{T} \in \mathcal{T}_n$ such that $f(\mathfrak{T}) \geq 0$ for every $f \in \mathcal{F}$.*

Proof. For $f \in \mathcal{F}$, let

$$B_f = \{T \in \mathcal{T}_n : f(T) \geq 0\} \subset \mathcal{T}_n.$$

By hypothesis, each B_f is nonempty and it suffices to prove that

$$\cap_{f \in \mathcal{F}} B_f \neq \emptyset.$$

Since each B_f is compact, it suffices to prove that the collection $\{B_f : f \in \mathcal{F}\}$ has the finite intersection property. Accordingly, let $f_1, \ldots, f_m \in \mathcal{F}$ be given. Arguing by contradiction, suppose

$$\cap_{j=1}^{m} B_{f_j} = \emptyset.$$

Define $F : \mathbb{S}_n \to \mathbb{R}^m$ by

$$F(T) = (f_1(T), \ldots, f_m(T)).$$

Then $F(\mathcal{T}_n)$ is both convex and compact because $\mathcal{T}_n$ is both convex and compact and each f_j, and hence F, is affine linear. Moreover, $F(\mathcal{T}_n)$ does not intersect

$$\mathbb{R}^m_+ = \{x = (x_1, \ldots, x_m) : x_j \geq 0 \text{ for each } j\}.$$

Hence there is a linear functional $\lambda : \mathbb{R}^m \to \mathbb{R}$ such that $\lambda(F(\mathcal{T}_n)) < 0$ and $\lambda(\mathbb{R}^m_+) \geq 0$. There exists λ_j such that $\lambda(x) = \sum \lambda_j x_j$. Since $\lambda(\mathbb{R}^m_+) \geq 0$, it follows that each $\lambda_j \geq 0$. Since $\lambda \neq 0$, for at least one k, $\lambda_k > 0$. Without loss of generality, it may be assumed that $\sum \lambda_j = 1$. Let

$$f = \sum \lambda_j f_j.$$

Since $\mathcal{F}$ is convex, it follows that $f \in \mathcal{F}$. On the other hand, $f(T) = \lambda(F(T))$. Hence if $T \in \mathcal{T}_n$, then $f(T) < 0$. Thus, for this f there does not exist a $T \in \mathcal{T}_n$ such that $f(T) \geq 0$, a contradiction that completes the proof. $\square$

LEMMA 6.2. *Let $\mathcal{C} = (\mathcal{C}(n))$ denote an open matrix convex subset of the graded set $\mathbb{S}(\mathbb{R}^g)$. Let n and a linear functional $\Lambda : \mathbb{S}_n(\mathbb{R}^g) \to \mathbb{R}$ be given. If $\Lambda(X) \leq 1$ for each $X \in \mathcal{C}(n)$, then there is a $\mathfrak{T} \in \mathcal{T}_n$ such that for each m, each $Y \in \mathcal{C}(m)$, and each $m \times n$ contraction (matrix) C,*

$$\Lambda(C^T Y C) \leq \operatorname{tr}(C \mathfrak{T} C^T).$$

Proof. Given a positive integer m, a tuple Y in $\mathcal{C}(m)$, and an $m \times n$ contraction matrix C, define $f_{Y,C} : \mathbb{S}_n \to \mathbb{R}$ by

$$f_{Y,C}(T) = \operatorname{tr}(C T C^T) - \Lambda(C^T Y C).$$

Now we show that the collection $\mathcal{F} = \{f_{Y,C} : Y, C\}$ is a convex set. Start with a positive integer s, nonnegative numbers $\lambda_1, \ldots, \lambda_s$ with $\sum \lambda_j = 1$, and with (Y_j, C_j) for $j = 1, \ldots, s$ where $Y_j \in \mathcal{C}(m_j)$ and C_j are $m_j \times n$ contraction matrices. Let $Z = \oplus Y_j$ and let F denote the (block) column matrix with entries $\sqrt{\lambda_j} C_j$. Then $Z \in \mathcal{C}(m)$, where $m = \sum m_j$ and

$$F^T F = \sum \lambda_j C_j^T C_j \preceq \sum \lambda_j I = I.$$

By definition,

$$\sum \lambda_j C_j^T Y_j C_j = F^T Z F$$

and

$$\sum \lambda_j \mathrm{tr}(C_j T C_j^T) = \mathrm{tr}(F T F^T).$$

Therefore,

$$\sum \lambda_j f_{Y_j, C_j}(T) = f_{Z,F}(T).$$

If C has (operator) norm one, choose $T = \gamma\gamma^T$ where γ is a unit vector such that

$$\|C\gamma\| = \|C\| = 1.$$

It follows that $\gamma\gamma^T \in \mathcal{T}_n$ and

$$f_{Y,C}(\gamma\gamma^T) = \|C\|^2 - \Lambda(C^T Y C) = 1 - \Lambda(C^T Y C).$$

Since $C^T Y C \in \mathcal{C}(n)$, the right-hand side above is nonnegative. If the contraction C does not have norm 1, but is not zero, a simple scaling argument shows that $f_{Y,C}(\gamma\gamma^T) \geq 0$ still. Consequently, for each $f_{Y,C}$ there is a $T \in \mathcal{T}_n$ such that $f_{Y,C}(T) \geq 0$. From Lemma 6.1, there is a $\mathfrak{T} \in \mathcal{T}_n$ such that $f_{Y,C}(\mathfrak{T}) \geq 0$ for every Y and C. $\square$

Given $\varepsilon > 0$, the *free ε-neighborhood of* 0, denoted $\mathcal{N}_\varepsilon$, is the graded set $(\mathcal{N}_\varepsilon(n))_{n=1}^\infty$ where

$$\mathcal{N}_\varepsilon(n) = \{X \in \mathbb{S}_n(\mathbb{R}^g) : \sum \|X_j\| < \varepsilon\}.$$

LEMMA 6.3. *If p satisfies the conditions of Assumption* 1.3, *then $\mathcal{D}_p$ contains an $\varepsilon > 0$ neighborhood of* 0; *i.e., there is an $\varepsilon > 0$ such that $\mathcal{N}_\varepsilon(n) \subset \mathcal{D}_p(n)$ for each n.*

Moreover, if the monic linear pencil $L = I + \sum A_j x_j$ is positive definite on $\mathcal{D}_p$, then $\|A_j\| \leq \frac{1}{\varepsilon}$ for each j.

Proof. Write p as in equation (1.5). Thus each p_w is a $\delta \times \delta$ matrix. Let M denote the maximum of $\{\|p_w\| : 1 \leq |w| \leq d\}$. Let $\tau = \sum_1^d g^j$. Thus τ is the number of words w with $1 \leq |w| \leq d$.

Let $0 < \Delta$ denote the minimum of $\{|\lambda| : \lambda \text{ is an eigenvalue of } p(0)\}$. Choose $\varepsilon = \min\{1, \frac{\Delta}{\tau(M+1)}\}$.

Let $X \in \mathbb{S}_n(\mathbb{R}^g)$ be given. If $\|X_j\| < \varepsilon$ for $1 \leq j \leq g$, then $\|w(tX)\| \leq \frac{\Delta}{\tau(M+1)}$ for nonempty words w and $0 \leq t \leq 1$. Hence,

$$\|\sum_{1 \leq |w| \leq d} p_w \otimes w(tX)\| \leq \sum_{1 \leq |w| \leq d} \|p_w\| \, \|w(tX)\| < \Delta.$$

It follows that $p(tX)$ is invertible for $0 \le t \le 1$ and thus $X \in \mathcal{D}_p(n)$. Consequently, $\mathcal{D}_p(n)$ contains $\mathcal{N}_\varepsilon(n)$.

Now suppose L is a monic linear pencil that is positive definite on $\mathcal{D}_p$ and thus on $\mathcal{N}_\varepsilon$. For $0 \le t < \varepsilon$, the points $\pm te_j$ are in $\mathcal{N}_\varepsilon(1)$ and hence $L(\pm te_j) = I \pm tA_j \succeq 0$. It follows that $\pm A_j \preceq \frac{1}{\varepsilon} I$ and thus $\|A_j\| \le \frac{1}{\varepsilon}$. $\square$

PROPOSITION 6.4. *Let $\mathcal{C} = (\mathcal{C}(n))$ denote a bounded open matrix convex subset of the graded set $\mathbb{S}(\mathbb{R}^g)$ that contains a free ε-neighborhood of* 0. *If $X^{\mathrm{b}} \in \mathbb{S}_n(\mathbb{R}^g)$ is in the boundary of $\mathcal{C}(n)$, then there is a monic linear pencil L (of size n) such that $L(Y) \succ 0$ for all m and $Y \in \mathcal{C}(m)$ and such that $L(X^{\mathrm{b}})$ is singular.*

Proof. By the usual Hahn-Banach Separation Theorem and the assumption that $\mathcal{C}(n)$ is open and contains an ε-neighborhood of 0, there is a linear functional $\Lambda : \mathbb{S}_n(\mathbb{R}^g) \to \mathbb{R}$ such that $\Lambda(X^{\mathrm{b}}) = 1 > \Lambda(\mathcal{C}(n))$.

From Lemma 6.2 there is a positive semi-definite $n \times n$ matrix T of trace one such that

$$\operatorname{tr}(CTC^T) - \Lambda(C^T Y C) \ge 0 \tag{6.1}$$

for each m, each $m \times n$ contraction C, and each $Y \in \mathcal{C}(m)$. Note this inequality is sharp in the sense that

$$\operatorname{tr}(T) - \Lambda(X^{\mathrm{b}}) = 0. \tag{6.2}$$

The rest of the proof amounts to expressing (6.1) in a concrete way in terms of a monic linear pencil.

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_g\}$ denote the standard orthonormal basis for $\mathbb{R}^g$. Given $1 \le \ell \le g$, define a bilinear form on $\mathbb{R}^n$ by

$$\mathcal{B}_\ell(c, d) = \frac{1}{2}\Lambda((cd^T + dc^T) \otimes \mathbf{e}_\ell)$$

for $c, d \in \mathbb{R}^n$. There is a unique real symmetric $n \times n$ matrix B_ℓ such that

$$\mathcal{B}_\ell(c, d) = \langle B_\ell c, d\rangle.$$

Let L_B denote the linear polynomial $L_B(x) = \sum_1^g B_j x_j$. Fix a positive integer m and let $\{e_1, \ldots, e_m\}$ denote the standard orthonormal basis for $\mathbb{R}^m$. Let $Y = (Y_1, \ldots, Y_g) \in \mathcal{C}(m)$ be given and consider $L_B(Y)$. Given a vector $\gamma = \sum_{j=1}^m \gamma_j \otimes e_j$ contained in $\mathbb{R}^n \otimes \mathbb{R}^m$, compute

$$\begin{aligned}
\langle L_B(Y)\gamma, \gamma\rangle =& \sum_{i,j} \sum_\ell \langle B_\ell \gamma_j, \gamma_i\rangle \langle Y_\ell e_j, e_i\rangle \\
=& \frac{1}{2} \sum_{i,j} \sum_\ell \Lambda((\gamma_j \gamma_i^T + \gamma_i \gamma_j^T) \otimes \mathbf{e}_\ell) \langle Y_\ell e_j, e_i\rangle \\
=& \Lambda(\sum_{i,j} \gamma_i (\sum_\ell \langle Y_\ell e_j, e_i\rangle \otimes \mathbf{e}_\ell) \gamma_j^T) \\
=& \Lambda(\Gamma Y \Gamma^T),
\end{aligned}$$

where Γ is the matrix with j-th column γ_j. Using equation (6.1)

$$\begin{aligned}\Lambda(\Gamma Y \Gamma^T) \leq &\mathrm{tr}(\Gamma^T T \Gamma)\\ =&\sum \langle T\gamma_j, \gamma_j\rangle\\ =&\sum \langle (T\otimes I)\sum_j \gamma_j \otimes e_j, \sum_k \gamma_k \otimes e_k\rangle\\ =&\langle (T\otimes I)\gamma, \gamma\rangle.\end{aligned}$$

Thus, the linear pencil $T - L_B$ defined by $(T - L_B)(x) = T - \sum B_j x_j$ satisfies

$$[T - L_B](Y) \succeq 0 \tag{6.3}$$

for every m and $Y \in \mathcal{C}(m)$.

Since $\mathcal{C}$ contains the ε-neighborhood of 0, it contains $\pm\frac{\varepsilon}{2}e_j \in \mathbb{R}^g$. Hence,

$$0 \preceq T - \pm\frac{\varepsilon}{2}L_B(e_j) = T - \pm\frac{\varepsilon}{2}B_j.$$

Thus, while T need not be invertible, it does satisfy $-T \preceq \frac{\varepsilon}{2}B_j \preceq T$ for each j and hence, restricting to the range of T (kernel of T^T), it can be assumed (passing to a space of smaller dimension if necessary) that T is invertible. Finally, multiplying left and right by $T^{-\frac{1}{2}}$ produces a linear polynomial $\mathcal{L}(x) = \sum_j A_j x_j$ such that $(I - \mathcal{L})(Y) \succeq 0$ if and only if $(T - L_B)(Y) \succeq 0$.

On the other hand, computing as above, (6.2) becomes

$$\langle (T - L_B)(X^{\mathrm{b}})e, e\rangle = 0 \quad \text{with} \quad e = \sum e_j \otimes e_j.$$

Since X^{b} is in the closure of $\mathcal{C}(n)$, $(T - L_B)(X^{\mathrm{b}}) \succeq 0$. Thus $(T - L_B)(X^{\mathrm{b}})e = 0$, and since $[T \otimes I]e \neq 0$, it follows that $(I - \mathcal{L})(X^{\mathrm{b}})$ is singular. Set $L = I - \mathcal{L}$.

Finally, the assumption that $\mathcal{C}$ is open implies that L is in fact positive definite, not just positive semi-definite, on $\mathcal{C}$. The proof of this statement is very similar to that of Lemma 2.2. The details are omitted. □

6.2. *Effros-Winkler and invertibility sets.* The following lemma is both a refinement and specialization of the free Hahn-Banach Separation Theorem of Effros and Winkler [EW97]. It is specialized to convex bounded sets $\mathcal{D}_p = (\mathcal{D}_p(n))$, and refined in that it separates a point on the boundary of $\mathcal{D}_p(m)$ from $\mathcal{D}_p$.

LEMMA 6.5. *Suppose p satisfies the conditions of Assumption* 1.3. *If $X \in \partial\mathcal{D}_p(m)$, then there exists a monic linear pencil L of size m such that L is positive definite on each $\mathcal{D}_p(n)$ and $L(X)$ is singular.*

Proof. By Theorem 5.5, $\mathcal{D}_p$ is a bounded open matrix convex set. By Lemma 6.3, $\mathcal{D}_p$ contains a free ε-neighborhood of 0. Hence an application of Proposition 6.4 proves the lemma. □

The following is a more quantitative version of Lemma 6.5. Recall $\nu = \delta\sum_0^d g^j$.

LEMMA 6.6. *Suppose p satisfies Assumption 1.3. If $(X,v) \in \widehat{\partial\mathcal{D}}_p(m)$, then there exists a monic linear pencil L of size $\ell \le \nu$, where ℓ is the dimension of*

$$\mathcal{M} = \mathcal{M}(X,v) = \{q(X)v : q \in \mathcal{P}_d^\delta\} \subset \mathbb{R}^m,$$

and a nonzero vector $w \in \mathbb{R}^\ell \otimes \mathcal{M}$ such that L is positive definite on each $\mathcal{D}_p(n)$ and $L(X)w = 0$.

Remark 6.7. In terms of $\{e_1,\dots,e_\ell\}$, the standard basis for $\mathbb{R}^\ell$, there exists $m_1,\dots,m_\ell \in \mathcal{M}$ such that $w = \sum e_\alpha \otimes m_\alpha$. From the definition of $\mathcal{M}$, there thus exists $q^\alpha \in \mathcal{P}_d^\delta$ such that $m_\alpha = q^\alpha(X)v$ and hence,

$$w = \sum e_\alpha \otimes q^\alpha(X)v.$$

Proof. Let $Y = P_{\mathcal{M}}X|_{\mathcal{M}}$. By Lemma 5.6, $(Y,v) \in \widehat{\partial\mathcal{D}}_p(\ell)$. By Lemma 6.5, there exists a monic linear pencil L of size ℓ such that L is positive definite on each $\mathcal{D}_p(n)$ and $L(Y)$ is singular. Hence there is a nonzero $w \in \mathbb{R}^\ell \otimes \mathcal{M}$ such that $L(Y)w = 0$. Since

$$\begin{aligned}\langle L(X)w,w\rangle =&\langle (I_\ell \otimes P_{\mathcal{M}})\ L(X)\ (I_\ell \otimes P_{\mathcal{M}})w,w\rangle\\ =&\langle L(Y)w,w\rangle\\ =&0,\end{aligned}$$

and since $L(X) \succeq 0$, the conclusion $L(X)w = 0$ follows. $\square$

6.3. *Dominating points and separation.* Proposition 6.8 relates dominating points to the separating monic linear pencils produced by Lemma 6.6. It is the main result of this section and the last ingredient needed for the proof of Theorem 1.4 in the next section.

Let $|w|$ denote the length of a word w. By convention, $|\emptyset| = 0$.

PROPOSITION 6.8. *Suppose p satisfies Assumption 1.3. If $S = (S(n))$ is a nonempty graded subset of the graded set $\widehat{\partial\mathcal{D}}_p$ that respects direct sums, then there exists a monic linear pencil L that is positive definite on each $\mathcal{D}_p(n)$ and singular on $S \cap S_* = (S(n) \cap S_*(n))$; that is, if $X \in \mathcal{D}_p(n)$, then $L(X) \succ 0$, and if $(X,v) \in S(n) \cap S_*(n)$, then $L(X)$ is singular. Further, the size of L can be chosen to be at most the maximum of the dimensions of the subspaces $\{q(Y)w : q \in \mathcal{P}_d^\delta\}$ over $(Y,w) \in S$ and is therefore at most dim $\mathcal{P}_d^\delta = \nu$.*

Proof. Let μ denote the maximum of the dimensions of the subspaces $\{q(Y)w : q \in \mathcal{P}_d^\delta\}$ for $(Y,w) \in S$.

Given $(X,v) \in S(m)$, let Λ_X denote the set of monic linear pencils L of size μ that are both positive definite on each $\mathcal{D}_p(n)$ and for which $L(X)$ is singular.

By identifying $L = I + \sum A_j x_j$ with the tuple $A = (A_1, \dots, A_g) \in \mathbb{S}_\mu(\mathbb{R}^g)$, the collection Λ_X may be viewed as a subset of a finite dimensional vector space.

By Lemma 6.6, each Λ_X is nonempty. By Lemma 6.3, each Λ_X is bounded. If a sequence from Λ_X converges to the monic linear pencil L, then $L(Y) \succeq 0$ for each n and $Y \in \mathcal{D}_p(n)$. By an application of Lemma 2.2, it follows that L is in fact positive definite on each $\mathcal{D}_p(n)$. Hence Λ_X is closed and thus compact.

Given an s and $(X^j, v^j) \in S(m_j) \cap S_*(m_j) \subset \widehat{\partial\mathcal{D}}_p(m_j)$ for $1 \le j \le s$, let $(W, u) = \oplus(X^j, v^j)$. Since S is closed with respect to direct sums, $(W, u) \in S(m)$, where $m = \sum m_j$.

Concordant with earlier usage, let

$$\mathcal{M}(W, u) := \ \{q(W)u : \ q \in \mathcal{P}_d^\delta \ \}.$$

By Lemma 6.6, there is a monic linear pencil $L = I + \sum A_j x_j$ of size μ such that L is positive definite on each $\mathcal{D}_p(n)$ and a nonzero vector $\gamma \in \mathbb{R}^\mu \otimes \mathcal{M}(W, u)$ such that $L(W)\gamma = 0$. From the definitions of $\mathcal{M}(W, u)$ and $\mathbb{R}^\mu \otimes \mathcal{M}(W, u)$, there exists $q^\alpha \in \mathcal{P}_d^\delta$ for $1 \le \alpha \le \mu$, such that

$$\gamma = \sum_{\alpha=1}^{\mu} e_\alpha \otimes q^\alpha(W)u.$$

Let

$$q = \sum_{\alpha=1}^{\mu} e_\alpha \otimes q^\alpha = \begin{pmatrix} q^1 \\ \vdots \\ q^\mu \end{pmatrix}.$$

Thus q is a $\mu \times \delta$ matrix of polynomials of degree at most d; that is, $q \in \mathcal{P}_d^{\mu \times \delta}$. Further,

$$\gamma = q(W)u.$$

Up to unitary equivalence (the canonical shuffle),

$$L(W)\gamma = L(W)q(W)u = \begin{pmatrix} L(X^1)q(X^1)v^1 \\ \vdots \\ L(X^s)q(X^s)v^\mu \end{pmatrix}.$$

Let

$$\gamma_j = q(X^j)v^j = \begin{pmatrix} q^1(X^j)v^j \\ q^2(X^j)v^j \\ \vdots \\ q^\mu(X^j)v^j \end{pmatrix}.$$

Since $L(W)\gamma = 0$,

$$L(X^j)\gamma_j = 0 \tag{6.4}$$

for each $1 \le j \le s$.

To prove that each $\gamma_j \neq 0$, we now invoke the hypothesis that each $(X^j, v^j) \in S(m_j) \cap S_*(m_j)$. If $\gamma_k = 0$ (for some k), then $q^\alpha(X^k)v^k = 0$ for each α. By Lemma 3.3, for a fixed α, either $q^\alpha(X^j)v^j = 0$ for every j or $q^\alpha(X^j)v^j \neq 0$ for every j. Since $q^\alpha(X^k)v^k = 0$, it follows that $q^\alpha(X^j)v^j = 0$ for every j and every α. Thus each $\gamma_j = 0$ and hence $\gamma = 0$, a contradiction.

Since, for each j, we have $\gamma_j \neq 0$, but $L(X^j)\gamma_j = 0$, it follows that $L \in \Lambda_{X^j}$. This proves

$$\cap_{j=1}^s \Lambda_{X^j} \neq \emptyset.$$

Consequently, the collection $\{\Lambda_X : (X, v) \in S(n) \cap S_*(n), \quad 1 \leq n\}$ of compact sets has the finite intersection property. Hence the full intersection is nonempty and any L in this intersection is positive definite on $\mathcal{D}_p$ and singular on all of $S(n) \cap S_*(n)$ for each n (meaning, for each n, if $(X, v) \in S(n) \cap S_*(n)$, then $L(X)$ is singular). □

Corollary 6.9. *If p satisfies Assumption 1.3, then the graded set $(\widehat{\partial\mathcal{D}_p})_* = (\widehat{\partial\mathcal{D}_p}(n)_*)$ is nonempty and there is a monic linear pencil L that is positive definite on $\mathcal{D}_p$ and singular on $(\widehat{\partial\mathcal{D}_p})_*$; that is, for each n, if $X \in \mathcal{D}_p(n)$ then $L(X) \succ 0$, and if $(X, v) \in (\widehat{\partial\mathcal{D}_p})_*(n)$, then $L(X)$ is singular.*

Proof. Note $\widehat{\partial\mathcal{D}_p} \cap (\widehat{\partial\mathcal{D}_p})_* = (\widehat{\partial\mathcal{D}_p})_*$ and apply Proposition 6.8 with $S = \widehat{\partial\mathcal{D}_p}$. □

7. Theorem 1.4

Theorem 1.4 is an immediate consequence of the following result.

Theorem 7.1. *Given p satisfying Assumption 1.3, there exists a monic linear pencil L such that L is positive definite on each $\mathcal{D}_p(n)$ and $L(X)$ has a kernel for every n and $X \in \partial\mathcal{D}_p(n)$. Hence the graded sets $\mathcal{D}_p = (\mathcal{D}_p(n))$ and $\mathcal{D}_L = (\mathcal{D}_L(n)) = (\{X \in \mathbb{S}_n(\mathbb{R}^g) : L(X) \succ 0\})$ are equal.*

Proof. Recall, for L, a monic linear pencil, $i(L) = (i(L)(n))$ is the graded set defined by

$$i(L)(n) := \ \{(Y, w) \in \widehat{\partial\mathcal{D}_p}(n) : \ L(Y) \text{ is invertible}\}.$$

We argue by contradiction. Accordingly, suppose for each monic linear pencil L that is positive definite on $\mathcal{D}_p$, the graded set $i(L)$ is nonempty.

Let $\mathfrak{S}$ denote pairs (S, L) with $S = (S(n))$ a $\mathcal{P}_d^\delta$-closed graded subset of the graded set $\widehat{\partial\mathcal{D}_p}$ and L a monic linear pencil satisfying

(i) L is positive definite on $\mathcal{D}_p$;
(ii) L is singular on S_*; and
(iii) $i(L) \subset S$.

Note that $\mathfrak{S}$ is not empty since, by Corollary 6.9, there is an L such that $(\widehat{\partial\mathcal{D}_p}, L) \in \mathfrak{S}$. Let $\mathfrak{S}_1$ denote the collection of graded sets S occurring in the pairs (S, L) belonging to $\mathfrak{S}$. Choose a minimal (with respect to term wise set inclusion) graded set S in $\mathfrak{S}_1$, whose existence is implied by Lemma 4.2(vii). We will show that S is not minimal, a contradiction that will complete the proof.

Since $S \in \mathfrak{S}_1$, there exists an L satisfying conditions (i), (ii), and (iii) with respect to this S; that is, $(S, L) \in \mathfrak{S}$. By assumption, $i(L)(k) \neq \emptyset$ for some k. By Proposition 4.5, $i(L)_z(m) \subsetneq S_z(m)$ for some m. Since also S is $\mathcal{P}_d^\delta$ closed ($S = S_z$),

$$i(L)_z \subsetneq S. \tag{7.1}$$

Using the fact that the graded set $i(L)$ is nonempty and respects direct sums, Proposition 6.8 produces a monic linear pencil M that is positive definite on each $\mathcal{D}_p(n)$ and singular on each $i(L)(n) \cap i(L)_*(n)$. The proof now proceeds by showing $(i(L)_z, L \oplus M) \in \mathfrak{S}$, which, by the strict inclusion in equation (7.1), contradicts the minimality of S.

From the construction, $L \oplus M$ is positive definite on each $\mathcal{D}_p(n)$; that is, $L \oplus M$ satisfies condition (i).

By Lemma 3.2, the graded set $i(L)_*$ is not empty. Suppose now that $(X, v) \in (i(L)_z)_*(n) = i(L)_*(n)$ (see Lemma 4.3(ii)). If $(X, v) \in i(L)(n)$, then $M(X)$, and hence $(L \oplus M)(X)$ is singular. On the other hand, if $(X, v) \notin i(L)(n)$, then $L(X)$, and hence $(L \oplus M)(X)$ is singular. Thus, if $(X, v) \in (i(L)_z)_*$, then $(L \oplus M)(X)$ is singular. Hence $L \oplus M$ satisfies condition (ii) with respect to $i(L)_z$.

Finally, for each n, $i(L \oplus M)(n) \subset i(L)(n) \subset i(L)_z(n)$ and thus $L \oplus M$ satisfies condition (iii) with respect to $i(L)_z$. Hence $(i(L)_z, L \oplus M) \in \mathfrak{S}$ and the proof is complete. □

7.1. *Estimates on the size of the linear pencil.* This subsection gives estimates on the size of the monic linear pencil L needed in Theorem 1.4. Recall $\nu = \delta \sum_0^d g^j$ is the dimension of $\mathcal{P}_d^\delta$.

LEMMA 7.2. *The size of L need in Theorem* 1.4 *is at most* $\frac{\nu(\nu+1)}{2}$.

Sketch of proof. The proof of Theorem 7.1 can be viewed as a recursive algorithm for constructing L as a direct sum $L = \oplus_{j=0}^k L_j$. The algorithm terminates in at most ν steps and, using the estimate afforded by Proposition 6.8, the dimension of L_j (its matrix size) at the j-th step is at most $\nu - j$. Thus $\frac{\nu(\nu+1)}{2}$ is an upper bound on the size of L. □

In the special case that $p(0) = p_\emptyset$ is positive definite, $\mathcal{D}_p(n)$ is equal to the component of 0 of the set $\{X \in \mathbb{S}_n(\mathbb{R}^g) : p(X) \text{ is positive definite}\}$ and accordingly, $\mathcal{D}_p$ is called the *positivity set* of p. In this case it can be assumed that $p(0) = I_\delta$. Moreover, the estimate on the size of L needed in Theorem 1.4 is reduced dramatically from that given in Proposition 7.2, because, as outlined below, the estimate of the size of the pencil in Proposition 6.8 can be reduced roughly by half.

Let $[\frac{d}{2}]_+$ denote the largest integer less than or equal to $\frac{d}{2}$. Let

$$\breve{\nu} = \delta \sum_{j=0}^{[\frac{d}{2}]_+} g^j. \tag{7.2}$$

Notice that $\breve{\nu}$ is the dimension of the vector space $\mathcal{P}^{\delta}_{[\frac{d}{2}]_+}$ and, given $(X, v) \in \widehat{\partial\mathcal{D}}_p$, it is thus an upper bound for the dimension of

$$\breve{M} = \{q(X)v : q \in \mathcal{P}^{\delta}_{[\frac{d}{2}]_+}\}.$$

The following lemma is a variant of Lemma 5.6, using the smaller space $\breve{M}$ instead of $\mathcal{M}$.

LEMMA 7.3. *Suppose $p \in \mathcal{P}^{\delta\times\delta}_d$ satisfies the conditions of Assumption* 1.3 *and moreover that $p(0) = I_\delta$. If $(X, v) \in \widehat{\partial\mathcal{D}}_p(n)$, then $(P_{\breve{M}}X|_{\breve{M}}, v) \in \widehat{\partial\mathcal{D}}_p(n)$; indeed, $tP_{\breve{M}}X|_{\breve{M}} \in \mathcal{D}_p(n)$ for $0 \le t < 1$ and $p(P_{\breve{M}}X|_{\breve{M}})v = 0$.*

Proof. Just as in Lemma 5.6, for $0 \le t < 1$, we have $tP_{\breve{M}}X|_{\breve{M}} \in \mathcal{D}_p$. Since $p(0) = I_\delta$, it follows that $p(tP_{\breve{M}}X|_{\breve{M}}) \succ 0$ and hence $p(P_{\breve{M}}X|_{\breve{M}}) \succeq 0$.

On the other hand, for any word w of length at most d, write $w = w_1x_jw_2$ where both words w_1 and w_2 have length at most $[\frac{d}{2}]_+$. Write $v \in \mathbb{R}^\delta \otimes \mathbb{R}^n$ as $v = \sum_{\alpha=1}^{\delta} e_\alpha \otimes v_\alpha$. Since both $w_2(X)v_\alpha$ and $w_1^T(X)v_\beta$ are in $\breve{M}$,

$$\begin{aligned}\langle w(P_{\breve{M}}X|_{\breve{M}})v_\alpha,\ v_\beta\rangle =& \langle P_{\breve{M}}X_jw_2(X)v_\alpha,\ w_1(X)^Tv_\beta\rangle\\ =& \langle X_jw_2(X)v_\alpha, w_1^T(X)v_\beta\rangle\\ =& \langle w(X)v_\alpha, v_\beta\rangle.\end{aligned}$$

Consequently,

$$\langle p(P_{\breve{M}}X|_{\breve{M}})v,\ v\rangle = \langle p(X)v,\ v\rangle = 0.$$

Since also $p(P_{\breve{M}}X|_{\breve{M}}) \succeq 0$, it follows that $p(P_{\breve{M}}X|_{\breve{M}})v = 0$. □

Applying Lemma 7.3 much like in the proof of Lemma 6.6 produces the following.

LEMMA 7.4. *Suppose p satisfies Assumption* 1.3 *and further that $p(0) = I_\delta$. If $(X, v) \in \widehat{\partial\mathcal{D}}_p(n)$, then there exists a monic linear pencil L of size $\ell \le \breve{\nu}$ and a nonzero vector $w \in \mathbb{R}^\ell \otimes \breve{M}$ such that L is positive definite on $\mathcal{D}_p$ and $L(X)w = 0$.*

Summarizing Lemma 7.2 and combining Lemma 7.4 with the argument behind Lemma 7.2 gives

Theorem 7.5. *Suppose p is a symmetric $\delta \times \delta$ matrix-polynomial of degree d in g variables that satisfies the conditions of Assumption* 1.3.

(i) *There is an $\ell \le \frac{\nu(\nu+1)}{2}$ and $\ell \times \ell$ symmetric matrices $A_1, \ldots, A_g$ such that $\mathcal{D}_p = \mathcal{D}_L$, where L is the monic linear pencil $L = I - \sum_j^g A_j x_j$.*

(ii) *In the case that $p(0) = I_\delta$, the estimate on the size of the matrices A_j in L reduces to $\frac{\breve{\nu}(\breve{\nu}+1)}{2}$, where $\breve{\nu} = \delta \sum_0^{[\frac{d}{2}]_+} g^j$.*

7.2. *Further remarks.*

Remark 7.6. We anticipate that the results of this paper remain valid if symmetric free variables are replaced by free variables; that is, with variables $(x_1, \ldots, x_g, y_1, \ldots, y_g)$ with the involution ^{T} on polynomials determined by $x_j^T = y_j$, $y_j^T = x_j$ and, for polynomials f and g in these variables, $(fg)^T = g^T f^T$. These polynomials are evaluated at tuples $X = (X_1, \ldots, X_g) \in M_n(\mathbb{R}^g)$ of $n \times n$ matrices with real entries. We do not see an obstruction to the free free variable analog of Theorem 1.4 using the arguments here. Indeed, arguments for such variables are often easier than for symmetric variables.

Remark 7.7. Fix a positive integer μ and let $\mathcal{L}$ denote a collection of monic linear pencils of size at most μ. The matrix convex set $\mathcal{C} = \mathcal{C}(n)$ defined by

$$\mathcal{C}(n) = \{X \in \mathbb{S}_n(\mathbb{R}^g) : L(X) \succ 0 \text{ for all } L \in \mathcal{L}\}$$

has the following finiteness property. If $X \in \mathbb{S}_n(\mathbb{R}^g)$, then $X \in \mathcal{C}(n)$ if and only if for every subspace $\mathcal{M}$ of $\mathbb{R}^n$ of dimension $k \le \mu$, the tuple $P_{\mathcal{M}} X|_{\mathcal{M}} \in \mathcal{C}(k)$. On the other hand, this latter property does not suffice to guarantee that $\mathcal{L}$ can be replaced by a finite collection of monic linear pencils. Thus, some additional hypothesis, such as assuming $\mathcal{D}_p$ is determined by a polynomial, is essential to reach the conclusion of Theorem 1.4.

8. The Case of irreducible p

The main result of this section, Theorem 8.3, says if p satisfies Assumption 1.3, $p(0) = I_\delta$, and p is irreducible in a sense made precise below, then p has degree at most two. Moreover, under these assumptions and with p scalar-valued ($\delta = 1$), Corollary 8.4 exhibits a very close connection between p and an L satisfying the conclusion of Theorem 1.4. Recall, p is a symmetric $\delta \times \delta$-matrix valued polynomial of degree d in g freely noncommuting variables.

Lemma 8.1. *Suppose $p \in \mathcal{P}_d^{\delta \times \delta}$ satisfies the conditions of Assumption* 1.3. *Suppose further that $p(0) = I_\delta$. If*

(i) $(X, v) \in \widehat{\partial \mathcal{D}}_p(n)$;

(ii) *L is a monic linear pencil of size ℓ that is positive definite on each $\mathcal{D}_p(n)$; and*

(iii) *there is a vector* $0 \neq w \in \mathbb{R}^\ell \otimes \breve{M}$, *where*

$$\breve{M} = \{q(X)v : q \in \mathcal{P}^\delta_{[\frac{d}{2}]_+}\},$$

such that $L(X)w = 0$,

then there exists a nonzero $q \in \mathcal{P}^\delta_{[\frac{d}{2}]_+ +1}$ *such that* $q(X)v = 0$. (*Note*: *it is not assumed that* L *is the* "*master monic linear pencil*" *from Theorem* 7.1.)

Proof. Write the monic linear pencil L as

$$L = I + \sum A_j x_j,$$

where the A_j are $\ell \times \ell$ symmetric matrices. The tuple X acts on $\mathbb{R}^n$, and hence $A_j \otimes X$ acts upon $\mathbb{R}^\ell \otimes \mathbb{R}^n$. With respect to this tensor product decomposition, $w = \sum e_j \otimes h_j$, where $\{e_1, \dots, e_\ell\}$ is the standard orthonormal basis for $\mathbb{R}^\ell$ and $h_j \in \breve{M}$. From the definition of $\breve{M}$, there exists polynomials $r_j \in \mathcal{P}^\delta_{[\frac{d}{2}]_+}$ such that $h_j = r_j(X)v$.

Since $L(X)w = 0$, for each m we have $0 = [e_m^T \otimes I]L(X)w$. Thus,

$$\begin{aligned} 0 =&[e_m^T \otimes I][w + \sum_k \sum_j A_k e_j \otimes X_k r_j(X)v] \\ =&[r_m + \sum_{k,j} (e_m^T A_k e_j) x_k r_j](X)v. \end{aligned}$$

Now we argue, by contradiction, that the elements q_m of $\mathcal{P}^\delta_{[\frac{d}{2}]_+ +1}$ given by

$$q_m(x) = r_m(x) + \sum_{k,j} (e_m^T A_k e_j) x_k r_j(x)$$

are not all 0. If they were all 0, then each r_m satisfies $r_m(0) = 0$; that is, r_m has no constant term. But then, by the same reasoning, each r_m has no linear terms, and continuing along these lines we ultimately conclude that all the r_m are 0. On the other hand, since $w \neq 0$, there is an m such that $h_m = r_m(X)v \neq 0$, a contradiction. Thus, there is an m such that $q_m \neq 0$ and at the same time $q_m(X)v = 0$. To complete the proof, observe that the degree of this q_m is at most $[\frac{d}{2}]_+ + 1$. □

Remark 8.2. Let R denote the element of $\mathcal{P}^{\ell \times \delta}$ whose m-th row is the r_m produced in the proof of Lemma 8.1. The lemma says that R is not zero. On the other hand, $R(X)v = w$ and $L(X)R(X)v = L(X)w = 0$. Hence the symmetric polynomial $R^T LR$ is nonzero, but vanishes at (X, v).

The polynomial p is a *minimum degree irreducible*, or *a minimum degree defining polynomial for* $\mathcal{D}_p$, provided the only $q \in \mathcal{P}^\delta_{d-1}$ that satisfies $q(X)v = 0$ for every n and every $(X, v) \in \widehat{\partial\mathcal{D}}_p(n)$ is $q = 0$. Note that, while p is restricted by Assumption 1.3 to be symmetric, the polynomial entries of q need not be symmetric. Of course, $q(X)$ is not symmetric (whenever $\delta > 1$), but rather an operator from $\mathbb{R}^\delta \otimes \mathbb{R}^n$ to $\mathbb{R}^n$.

THEOREM 8.3. *If the polynomial* $p \in \mathcal{P}_d^{\delta\times\delta}$ *satisfies Assumption* 1.3 *and if also* $p(0) = I_\delta$, *then there exists a nonzero* $q \in \mathcal{P}^\delta_{[\frac{d}{2}]_+ +1}$ *such that* $q(X)v = 0$ *for every* n *and* $(X, v) \in \widehat{\partial\mathcal{D}}_p(n)$.

In particular, if the graded set $\mathcal{D}_p = (\mathcal{D}_p(n))$ *is bounded and convex, if* $p(0) = I_\delta$, *and if* p *is a minimum degree defining polynomial for* $\mathcal{D}_p$, *then the degree of* p *is at most two.*

Proof. Given $(X, v) \in \widehat{\partial\mathcal{D}}_p(n)$, let

$$C_{(X,v)} = \{q \in \mathcal{P}^\delta_{[\frac{d}{2}]_+ +1} : \quad q(X)v = 0\}.$$

Note that $C_{(X,v)}$ is a subspace of $\mathcal{P}^\delta_{[\frac{d}{2}]_+ +1}$.

Let $\check{M} = \{r(X)v : r \in \mathcal{P}^\delta_{[\frac{d}{2}]_+}\}$. By Proposition 7.4 there is a monic linear pencil L of some size $\ell \leq \check{\nu}$ ($\check{\nu}$ is defined in Equation (7.2)) such that L is positive definite on $\mathcal{D}_p$ and a nonzero vector $w \in \mathbb{R}^\ell \otimes \check{M}$ such that $L(X)w = 0$. Thus Lemma 8.1 applies to produce a nonzero $q \in \mathcal{P}^\delta_{[\frac{d}{2}]_+ +1}$ such that $q(X)v = 0$. Hence $C_{(X,v)}$ is nontrivial (not (0)).

Given a positive integer s and $(X^j, v^j) \in \widehat{\partial\mathcal{D}}_p(m_j)$ for $1 \leq j \leq s$, let $(W, u) = \oplus(X^j, v^j)$. Then $(W, u) \in \widehat{\partial\mathcal{D}}_p(m)$, where $m = \sum m_j$. Further, by what has already been proved, there exists a nonzero $q \in \mathcal{P}^\delta_{[\frac{d}{2}]_+ +1}$ such that $q(W)u = 0$. But then $q(X^j)v^j = 0$ for each j. Hence $q \in \cap_{j=1}^\ell C_{(X^j,v^j)}$. It follows that the collection of subspaces $C_{(X,v)}$ is closed with respect to finite intersections. Since also each $C_{(X,v)}$ is a nontrivial subspace of the finite dimensional space $\mathcal{P}^\delta_{[\frac{d}{2}]_+ +1}$, there is a smallest (and nontrivial) subspace $C_{(Y,w)}$ uniquely determined by the condition that it has minimum dimension. Note that any (nonzero) $q \in C_{(Y,w)}$ must vanish on all of $\widehat{\partial\mathcal{D}}_p$, since if $(X, v) \in \widehat{\partial\mathcal{D}}_p$ and $q(X)v \neq 0$, then $C_{(X,v)} \cap C_{(Y,w)} \subsetneq C_{(Y,w)}$.

The second part of the theorem follows immediately from the first part and the definition of minimum degree defining polynomial. $\square$

COROLLARY 8.4. *Suppose* $p \in \mathcal{P}_d^{\delta\times\delta}$ *satisfies the conditions of Assumption* 1.3, $p(0) = I_\delta$, *and* p *is a minimum degree defining polynomial for* $\mathcal{D}_p$. *If* $\delta = 1$, *there exists a* 1×1 *monic linear pencil* L_0, *an integer* $m \leq g$, *and an* $m \times 1$ *linear pencil* $\hat{L}$ *with* $\hat{L}(0) = 0$ *such that* $\mathcal{D}_p = \mathcal{D}_L$, *where*

$$L = \begin{pmatrix} I_m & \hat{L} \\ \hat{L}^T & L_0 \end{pmatrix}.$$

In fact, p *is the Schur complement of the* $(1, 1)$ *entry of* L; *i.e.,*

$$p = L_0 - \hat{L}^T\hat{L}.$$

This corollary of Theorem 8.3 is, for the most part, an improvement over the main result of [DHM07]. In particular, the result here removes numerous hypotheses found in [DHM07] while reaching a stronger conclusion, though here it is assumed that $\mathcal{D}_p$ is convex, rather than the weaker condition that $\overline{\mathcal{D}}_p$ is convex. The techniques here are completely different from those in [DHM07].

Proof. The first part of Corollary 8.4 is covered by Theorem 8.3. It remains to prove that if p is a symmetric free polynomial in $\mathcal{P}_2^{1\times 1}$, if $p(0) = 1$, and if $\mathcal{D}_p$ is both bounded and convex, then p has the form

$$p = 1 + \ell(x) - \sum_{j=1}^{g} \lambda_j(x)^2,$$

where ℓ and each λ_j are linear.

Since p has degree two and is symmetric, there is a uniquely determined symmetric $g \times g$ matrix Λ such that

$$p(x) = 1 + \ell(x) - \langle \Lambda x, x\rangle,$$

where x is the vector with entries x_j. If Λ is not positive semi-definite, then there is a $t \in \mathbb{R}^g$ such that $\langle \Lambda t, t\rangle < 0$ and hence, for $s \in \mathbb{R}$,

$$p(st) = 1 + s\ell(t) - s^2\langle \Lambda t, t\rangle$$

is either positive for all $s \geq 0$ or is positive for all $s \leq 0$ depending upon the sign of $\ell(t)$. In either case, $\mathcal{D}_p(1)$ is not bounded. Thus the boundedness of $\mathcal{D}_p$ implies that Λ is positive semi-definite. Hence there is an $0 \leq m \leq g$ and an orthogonal set of vectors $u_1, \dots, u_g$ such that

$$\Lambda = \sum_1^m u_\ell u_\ell^T.$$

Letting $\lambda_\ell = \sum_j (u_\ell)_j x_j$,

$$\hat{L} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix},$$

and $L_0 = 1 + \ell$ the conclusion of the corollary follows. □

The following example shows that Corollary 8.4 requires the irreducibility hypothesis. Here we work with two variables (x, y). Let $b(x, y) = 1 - x^2 - y^2$ and $f(x, y) = 1 - (x - \frac{1}{4})^2 - y^2$. The set

$$\mathcal{D} = \mathcal{D}_{b\oplus f} = \{(X, Y) : b(X, Y) \succ 0,\ f(X, Y) \succ 0\}$$

is convex. Let $p_1 = fbf$ and $p_2 = bfb$. Then $\mathcal{D}_{p_1} = \mathcal{D} = \mathcal{D}_{p_2}$. Hence neither p_1 nor p_2 is a minimum degree defining polynomial for $\mathcal{D}$. Indeed, bf vanishes on $\widehat{\partial\mathcal{D}}_{p_1}$ and fb on $\widehat{\partial\mathcal{D}}_{p_2}$. On the other hand, neither bf nor fb is a symmetric so neither is a candidate for a minimum degree defining polynomial. It is likely that in this example there does not exist a minimum degree defining polynomial for $\mathcal{D}$.

9. **Free real algebraic geometry**

One of the main branches of real algebraic geometry, dating back to Hilbert, is semi-algebraic geometry, a subject that deals with polynomial inequalities. Free (noncommutative) semi-algebraic geometry has been developing for about a decade.

This section describes implications of the LMI representation of Theorem 1.4 for free semi-algebraic geometry. It also contains a strengthening of Theorem 1.4. Another area of contact is semi-definite programming (SDP), one of the main developments in optimization over the last two decades. We state one (disturbing) result in the language of SDP in Section 9.6.

9.1. *Free semi-algebraic sets.* This subsection gives definitions of free semi-algebraic sets and their principal components. Recall, from Section 1.4, that $pc[\mathcal{W}]$ denotes the *principal component* of a graded set $\mathcal{W} \subset \mathbb{S}(\mathbb{R}^g)$. Also, if p is a matrix-valued symmetric polynomial and $p(0)$ is invertible, then $pc[\mathfrak{I}_p] = \mathcal{D}_p$.

LEMMA 9.1. *If $p_j \in \mathcal{P}^{\delta_j \times \delta_j}$ is symmetric and $p_j(0)$ is invertible for $j = 1, 2, \ldots, s$, then*

$$\cap_1^s \mathfrak{I}_{p_j} = \mathfrak{I}_p \qquad \text{and} \qquad \cap_1^s \mathcal{D}_{p_j} \supset \mathcal{D}_p, \tag{9.1}$$

where $p = \oplus p_j$. Further

$$pc[\cap_1^s \mathcal{D}_{p_j}] = \mathcal{D}_p. \tag{9.2}$$

Proof. The intersection property of $\mathfrak{I}$ is obvious, as is the inclusion, $\mathfrak{I}_p \subset \mathfrak{I}_{p_j}$, for each j. Hence $\mathcal{D}_p = pc[\mathfrak{I}_p] \subset pc[\mathfrak{I}_{p_j}] = \mathcal{D}_{p_j}$, so $\cap_1^s \mathcal{D}_{p_j} \supset \mathcal{D}_p$ and

$$pc[\cap_1^s \mathcal{D}_{p_j}] \supset \mathcal{D}_p.$$

Since $\mathcal{D}_{p_j} \subset \mathfrak{I}_{p_j}$, we have $\cap_1^s \mathcal{D}_{p_j} \subset \cap_1^s \mathfrak{I}_{p_j} = \mathfrak{I}_p$. Consequently,

$$pc[\cap_1^s \mathcal{D}_{p_j}] \subset pc[\cap_1^s \mathfrak{I}_{p_j}] = pc[\mathfrak{I}_p] = \mathcal{D}_p. \qquad \square$$

Classically, a basic open semi-algebraic set is a set of the form

$$S = \{x \in \mathbb{R}^g : \mathsf{q}_j(x) > 0, \quad j = 1, \ldots, \sigma\}$$

for given (commutative) polynomials q_j [BCR98]. There are several natural ways to extend this definition to free $*$-algebras. The one that follows has the property that theorems flowing from it are stronger than analogous theorems using other definitions. Paralleling classical real algebraic geometry, we define a *free basic open semi-algebraic set* (containing 0) to be a graded set of the form $\cap_j \mathcal{D}_{p_j}$ for some finite set of symmetric matrix polynomials p_j in $\mathcal{P}^{\delta_j \times \delta_j}$ with $p_j(0)$ invertible. Note, while each $\mathcal{D}_{p_j}$ is a connected set, the intersection need not be. A *free open semi-algebraic set* (containing 0) is a finite union of free basic open semi-algebraic sets.

In classical real algebraic geometry, the components of a semi-algebraic set are themselves semi-algebraic. Lemma 9.1 says that the component of 0 of a free basic open semi-algebraic set is again free basic open semi-algebraic, and Corollary 9.3 gives natural conditions under which the principal component of a free open semi-algebraic set is itself a free basic open semi-algebraic set.

This section develops some properties of free open semi-algebraic sets, several of which contrast markedly with the classical situation. These properties lead to a strengthening of Theorem 1.4, and they are used to show that if the projection of an LMI representable set is a free open semi-algebraic set, then it is in fact LMI representable.

9.2. *Connectedness*. Before turning to free semi-algebraic sets, this subsection derives some fairly general facts with the theme of connectedness.

Proposition 9.2. *Suppose $p_j \in \mathcal{P}^{\delta_j\times\delta_j}$ for $j = 1, 2, \ldots, s$, each p_j is symmetric, and each $p_j(0)$ is invertible. Further, suppose $\mathcal{W} = (\mathcal{W}(n))$ is a graded set with $\mathcal{W}(n) \subset \cup_{j=1}^{s}\mathfrak{I}_{p_j}(n)$ for each n; that is, $\mathcal{W} \subset \cup_1^s\mathfrak{I}_{p_j}$. If $\mathcal{W}$ respects direct sums and each $\mathcal{W}(n)$ contains 0 and is open and connected, then there is a k such that $\mathcal{W} \subset \mathcal{D}_{p_k}$; that is, $\mathcal{W}(n) \subset \mathcal{D}_{p_k}(n)$ for each n.*

Proof. We begin by proving if $X \in \mathcal{W}(n)$ and if $X(t)$ is a (continuous) path for $0 \le t \le 1$ such that $X(0) = 0$, $X(1) = X$, and $X(t)$ lies in $\mathcal{W}(n)$, then there is a j such that $p_j(X(t))$ is invertible for every $0 \le t \le 1$.

Arguing by contradiction, suppose no such j exists. Then for every $1 \le \ell \le N$, there exists a $0 \le t_\ell \le 1$ such that $p_\ell(X(t_\ell))$ is not invertible. Since $\mathcal{W}$ is closed with respect to direct sums, $Z = \oplus X(t_\ell) \in \mathcal{W}(nN)$. It follows that there is some $1 \le j \le N$ such that $Z \in \mathfrak{I}_{p_j}(nN)$ and in particular, $p_j(Z)$ is invertible, contradicting $p_j(X(t_j))$ not invertible. We conclude that there is some j such that $p_j(X(t))$ is invertible for $0 \le t \le 1$ and hence $X(t) \in \mathcal{D}_{p_j}$ for all $0 \le t \le 1$.

Now suppose there is an m and a $Y \in \mathcal{W}(m)$ such that $Y \notin \mathfrak{I}_{p_s}(m)$. In particular, $p_s(Y)$ is not invertible. Since Y is in $\mathcal{W}(m)$, there is a continuous path $Y(t) \in \mathcal{W}(m)$ such that $Y(0) = 0$ and $Y(1) = Y$. Now let n and $X \in \mathcal{W}(n)$ be given. There is a continuous path $X(t) \in \mathcal{W}(n)$ with $X(0) = 0$ and $X(1) = X$. Let $Z(t) = X(t) \oplus Y(t)$, which is in $\mathcal{W}(n+m)$ since $\mathcal{W}$ respects direct sums. Thus $Z(t) \in \mathcal{W}(n+m)$ is a continuous path $(0 \le t \le 1)$ with $Z(0) = 0$. From what has already been proved, there is a j such that $p_j(Z(t))$ is invertible for each $0 \le t \le 1$. Thus $p_j(Y)$ is invertible and we conclude that $j \ne s$. At the same time, $p_j(X(t))$ is invertible for $0 \le t \le 1$ and thus $X \in \mathcal{D}_{p_j}$. Hence $X \in \cup_1^{s-1}\mathcal{D}_{p_j}(n)$. We have proved: either $\mathcal{W}(m) \subset \mathfrak{I}_{p_s}(m)$ for every m, or $\mathcal{W}(n) \subset \cup_1^{s-1}\mathcal{D}_{p_j}(n) \subset \cup_1^{s-1}\mathfrak{I}_{p_j}(n)$ for every n. Since $\mathcal{W}$ is connected and contains 0, the first alternative becomes $\mathcal{W}$ is a subset of $\mathcal{D}_{p_s}$; that is, $\mathcal{W}(m) \subset \mathcal{D}_{p_s}(m)$ for every m. Induction now finishes the proof. □

COROLLARY 9.3. *Suppose $p_{k,j}$ is a finite collection ($k = 1, \dots, t$; $j = 1, \dots, s_k$) of symmetric matrix-valued free polynomials with $p_{k,j}(0)$ invertible. Suppose the graded set $\mathcal{W} = (\mathcal{W}(n))$ has the form*

(i) $\mathcal{W} = pc[\cup_{k=1}^{t} \cap_{j=1}^{s_k} \mathcal{D}_{p_{k,j}}]$; *or*
(ii) $\mathcal{W} = pc[\cup_{k=1}^{t} \cap_{j=1}^{s_k} \mathfrak{I}_{p_{k,j}}]$.

If $\mathcal{W}$ respects direct sums, then there is a k_0 such that $\mathcal{W} = \mathcal{D}_{p^{k_0}}$, where p^k is defined by

$$p^k = \oplus_{j=1}^{s_k} p_{k,j}.$$

Proof. Either of the hypotheses (i) or (ii) imply that

$$\mathcal{W} \subset \cup_{k=1}^{t} \cap_{j=1}^{s_k} \mathfrak{I}_{p_{k,j}} = \cup_{k=1}^{t} \mathfrak{I}_{p^k},$$

the equality holding because of Lemma 9.1. Proposition 9.2 implies there is a k_0 such that $\mathcal{D}_{p^{k_0}} \supset \mathcal{W}$. Because of this containment, hypothesis (i), and Lemma 9.1, we have

$$\mathcal{D}_{p^{k_0}} \supset \mathcal{W} = pc[\cup_{k=1}^{t} \cap_{j=1}^{s_k} \mathcal{D}_{p_{k,j}}] \supset pc[\cup_{k=1}^{t} \mathcal{D}_{p^k}] \supset \mathcal{D}_{p^{k_0}}.$$

Thus $\mathcal{W} = \mathcal{D}_{p^{k_0}}$. Hypothesis (ii) along with $\mathfrak{I}_p \supset \mathcal{D}_p$ and $\mathcal{D}_{p^{k_0}} \supset \mathcal{W}$ imply hypothesis (i) holds. So, under either hypothesis, the required conclusion follows. □

9.3. *Free semi-algebraic sets vs. basic ones.* Corollary 9.3(i) rephrased in terms of semi-algebraic sets gives the following result.

COROLLARY 9.4. *Let $\mathcal{W} = (\mathcal{W}(n))$ be a graded set that is contained in* (*resp. is the principal component of*) *a free open semi-algebraic set. If $\mathcal{W}$ respects direct sums and each $\mathcal{W}(n)$ contains* 0 *and is open and connected, then $\mathcal{W}$ is contained in* (*resp. equals*) *the principal component of some free basic open semi-algebraic set.*

Theorem 1.4 can now be strengthened as follows.

THEOREM 9.5. *Suppose the graded set $\mathcal{W}$ is bounded and matrix convex.*

(i) *If $\mathcal{W}$ is the principal component of a free open semi-algebraic set*; *or*
(ii) *if $\mathcal{W}$ is the principal component of a graded set of the form*

$$\cup_{k=1}^{t} \cap_{j=1}^{s_k} \mathfrak{I}_{p_{k,j}},$$

then $\mathcal{W}$ has an LMI *representation.*

Proof. Because $\mathcal{W}$ is matrix convex, it is closed with respect to direct sums. Thus, under either hypothesis (i) or (ii) , Corollary 9.3 implies that $\mathcal{W}$

has the form $\mathcal{D}_p$ for some symmetric matrix-valued p. Further, $\mathcal{D}_p$ is convex, and thus Theorem 1.4 implies that $\mathcal{D}_p$ has an LMI representation. □

This theorem implies that the principal component of a free open semi-algebraic set is itself free semi-algebraic under the additional hypothesis that it is matrix convex.

9.4. *Free projections.* One of the key facts in real algebraic geometry is that the projection of a semi-algebraic set is again semi-algebraic (by Tarski's principle). Thus, if $S \subset \mathbb{R}^{g+h}$ is an open semi-algebraic set, then the projection onto its first g coordinates is a semi-algebraic set. Given a graded subset $\mathcal{D} = (\mathcal{D}(n))$ of the graded set $\mathbb{S}(\mathbb{R}^{g+h})$, the (*free*) *projection* of $\mathcal{D}$ (onto $(\mathbb{S}_n(\mathbb{R}^g))$) is the graded set $\pi(\mathcal{D}) = (\pi(\mathcal{D})(n))$ defined by

$$\pi(\mathcal{D}(n)) = \{X \in \mathbb{S}_n(\mathbb{R}^g) : \text{ there is a } Y \in \mathbb{S}_n(\mathbb{R}^h) \text{ such that } (X,Y) \in \mathcal{D}(n)\}.$$

Lemma 9.6. *The following properties are inherited under free projections.*

(i) *respects direct sums*;
(ii) *respects unitary conjugation*;
(iii) *openness*;
(iv) *connectedness*;
(v) *boundedness*; *and*
(vi) *matrix convexity.*

Proof. Straightforward. □

An immediate consequence of combining this lemma with Theorem 9.5(i) is a fact that is far from what one finds in the classical commutative case.

Corollary 9.7. *If the graded subset $\mathcal{W}$ of $\mathbb{S}(\mathbb{R}^{g+h})$ is bounded and has an representation and if its projection $\pi(\mathcal{W})$ is a free open semi-algebraic set, then $\pi(\mathcal{W})$ has an representation.*

This corollary plus the example in the following subsection shows that the projection of a free bounded basic open semi-algebraic set need not be free open semi-algebraic. We state this as a proposition, since it is so contrary to a basic tenet of classical real algebraic geometry.

Proposition 9.8. *There exists a monic linear pencil L in $g+h$ variables such that the projection $\pi(\mathcal{D}_L)$ is neither of the form* (i) *nor* (ii) *in Theorem* 9.5. *In particular, there exist convex free basic open semi-algebraic sets with projections that are not free open semi-algebraic.*

To establish the proposition, it suffices (thanks to Corollary 9.7) to produce a monic linear pencil L in $g+h$ variables with the property that $\pi(\mathcal{D}_L)$ is not of the form $\mathcal{D}_M$ for a monic linear pencil M in g variables. The following is an example of such an L.

9.5. *The TV screen*: *an example*. Consider the set

$$S = \{(x_1, x_2) \in \mathbb{R}^2 : 1 - x_1^4 - x_2^4 > 0\},$$

often called the TV screen. This set is evidently convex. By the line test in [HV07], there does not exist a monic linear pencil L such that $\mathcal{D}_L(1) = S$. Thus, *if* $\mathcal{T} = (\mathcal{T}(n))$ *is a graded set with* $\mathcal{T}(1) = S$, *then* $\mathcal{T}$ *does not have an* LMI *representation.*

Now we build a certain type of representation for S. Given α a positive real number, choose $\gamma^4 = 1 + 2\alpha^2$ and let

$$L_0^\alpha = \begin{pmatrix} 1 & 0 & y_1 \\ 0 & 1 & y_2 \\ y_1 & y_2 & 1 - 2\alpha(y_1 + y_2) \end{pmatrix} \tag{9.3}$$

and

$$L_j^\alpha = \begin{pmatrix} 1 & \gamma x_j \\ \gamma x_j & \alpha + y_j \end{pmatrix}, \quad j = 1, 2. \tag{9.4}$$

Note that the L_j^α are not monic, but because $L_j^\alpha(0) \succ 0$, they can be normalized to be monic without altering the solution sets of $L_j^\alpha(X) \succ 0$. The fact that

$$S = \{(x_1, x_2) \in \mathbb{R}^2 : \text{there exists } (y_1, y_2) \text{ such that } L_j^\alpha(x, y) \succ 0, \quad j = 0, 1, 2\}$$

follows by taking Schur complements and a bit of algebra, which shows

$$S = \{(x_1, x_2) : \ 1 - 2\alpha(y_1 + y_2) - y_1^2 - y_2^2 > 0, \quad \alpha + y_j > \sqrt{1 + 2\alpha^2}\, x_j^2\}.$$

Consequently, $S = \pi(\mathcal{D}_{L^\alpha}(1))$, *where* $L^\alpha = L_0^\alpha \oplus L_1^\alpha \oplus L_2^\alpha$.

Note that choosing $\alpha = 0$ gives the representation of the TV screen S often found in the literature. It is not satisfactory for the present purposes, since it can not be normalized to be monic.

Proof of Proposition 9.8. As was seen in the example above, for $\alpha > 0$ fixed, $\pi(\mathcal{D}_{L^\alpha})$ is the projection of an LMI representable set. However, it does not have either of the forms (i) or (ii) given in the proposition, as otherwise it would have, by Theorem 9.5, an LMI representation, thereby contradicting paragraph one of the example. ☐

9.6. *Outside perspectives*. Here we include two remarks aimed at readers with interest in either semi-definite programming or free real algebraic geometry.

Remark 9.9. The paradigm problem in semi-definite programming is to maximize a linear functional over an SDP representable set. A subset $C \subset \mathbb{R}^g$ is called semi-definite programming representable or *SDP representable*, if there is a monic linear pencil L in $g+h$ variables such that $C = \pi(\mathcal{D}_L(1))$. For a general survey and overview of semi-definite programming, see Nemirovski's Plenary Lecture at the 2006 ICM [Nem07].

By analogy with the scalar commutative case, a graded subset $\mathcal{C} = (\mathcal{C}(n))$ of $\mathbb{S}(\mathbb{R}^g)$ is *freely SDP representable* if there is a monic linear pencil L in $g+h$ variables such that $\mathcal{C}(n) = \pi(\mathcal{D}_L(n))$ for each n. For example, the graded set $\pi(\mathcal{D}_{L^\alpha})$ has, by construction, a free SDP representation. In this terminology, Corollary 9.7 says

> *if $\mathcal{C}$ is both* SDP *representable and free semi-algebraic, then $\mathcal{C}$ is* LMI *representable.*

Remark 9.10. As mentioned earlier, there are several other natural choices of the notion of free semi-algebraic set beyond the one adopted earlier. Here we mention one. Given a symmetric $p \in \mathcal{P}^{\delta\times\delta}$ with $p(0) \succ 0$ (not just invertible), let

$$\mathfrak{P}_p(n) = \{X \in \mathbb{S}_n(\mathbb{R}^g) : p(X) \succ 0\}.$$

Observe that $\cap \mathfrak{P}_{p_j} = \mathfrak{P}_p$, where $p = \oplus p_j$. The lemmas and theorems of this section, appropriately modified, hold if $\mathfrak{P}_p$ is used as the notion of a free basic open semi-algebraic set. For example, if $pc[\cup_{k=1}^s \mathfrak{P}_{q_k}]$ is bounded and matrix convex, then it has an LMI representation and is thus a free basic open semi-algebraic set.

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